Computerized Design, Generation, and Simulation of Meshing and Contact of Face-Milled Formate Cut Spiral Bevel Gears

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Abstract

A new approach for design, generation, and computerized simulation of meshing and contact of face-milled, formate cut spiral bevel gears is presented. The purpose is to develop a low noise, stabilized bearing contact for this type of gear drives. The approach proposed is based on application of three procedures that permit in sequence, to provide a longitudinally directed bearing contact, a predesigned parabolic function of transmission errors and limit the shift of bearing contact caused by errors of alignment. The theory developed is illustrated with an example of design and computation.

Nomenclature

- $\alpha_g$: Profile angle of parabolic blade for gear head cutter at mean point M
- $\alpha_p$: Blade angle for straight blade of pinion head cutter
- $\delta$: Elastic approach of contacting surfaces
- $\gamma_i$, $(i=1,2)$: Pitch cone angles of pinion $(i=1)$ and gear $(i=2)$, respectively
- $\gamma_{m_i}$, $(i=1,2)$: Machine root angles for pinion $(i=1)$ and gear $(i=2)$, respectively
- $\gamma_c$, $(i=1,2)$: Root cone angles for pinion $(i=1)$ and gear $(i=2)$, respectively
- $\gamma$: Shaft angle
- $\eta_i$, $(i=1,2)$: Tangent to the path of contact on the pinion $(i=1)$ and gear $(i=2)$, respectively
- $\theta_p$, $\theta_g$: Surface parameters of the pinion and gear head cutters, respectively
- $\lambda_f$, $\lambda_w$: Surface parameters of the pinion and the gear fillet parts of the head cutter
- $\sigma^{(i)}$: Angle formed between principal directions $(i, j = 1, 2, p, g)$
- $\phi_i$, $(i=1,2)$: Angles of rotation of pinion $(i=1)$ and gear $(i=2)$ in the process of meshing
- $\psi_{ci}$: Angle of rotation of the cradle in the process for generation of the pinion
- $\psi_r$: Angle of rotation of the pinion in the process for generation
- $\rho_f$, $\rho_w$: Fillet radii for the pinion and gear
- $\omega^{(1)}$: Angular velocity of the pinion (in meshing and generation)
- $\Sigma_i$, $(i=1,2)$: Pinion $(i=1)$ or gear $(i=2)$ tooth surface
- $\Sigma_k$, $(k=p,g)$: Pinion $(k=p)$ or gear $(k=g)$ generating surface
- $\Delta H$, $\Delta D$: Pinion and gear axial displacements, respectively
- $E_{m_i}$: Blank offset for pinion
- $X_{B_i}$: Sliding base for pinion
- $X_{D_i}$: Machine center to back for pinion
- $X_G$: Machine center to back for gear
ΔE, Δγ
Errors of the offset and shaft angle, respectively

Δφ₂ (φ₁)
Function of transmission errors

αₖ
Parabola coefficient of gear head cutter blade

α₉₃
Dedendum of pinion

Aₘ, F_w
Mean cone distance and face width

bₙ
Coefficients of modified roll for pinion generation, (i=1, 2, 3)

e₁, e₂, e₃, e₄
Unit vectors of principal directions of pinion and gear tooth surfaces, respectively

e₁, e₂, e₃, e₄
Unit vectors of principal directions of cutter surfaces for pinion and gear, respectively

kₙ(i)
Normal curvature of surface Σᵢ

kₙ(i)
Relative normal curvature

kᵣ, kₛ, kₐ, kₜ
Principal curvatures of pinion and gear tooth surfaces, respectively

kᵣ, kₛ, kₐ, kₜ
Principal curvatures of cutter surfaces for pinion and gear, respectively

Lₘₐ, Rₘₐ
Cylindrical coordinates for contact point A on gear tooth surface

m₁₂
Second derivative of transmission function φ₂ (φ₁)

m₁₂
Gear ratio

M
Mean contact point

M₁ᵦ, Lₖᵦ
Matrix of coordinate transformation from system Sᵢ to system Sⱼ

nᵾ(i), Nᵾ(i)
Unit normal and normal to the surface Σᵢ represented in coordinate system Sᵢ

nᵾ(i)
Tooth number of pinion (i=1) or gear (i=2)

φ₄(i)
Installment angle for pinion head cutter

Rₚ, Rₔ
Radii of the head-cutter at mean point for the pinion and gear

Rₚ
Gear cutter radii

rᵢ
Position vector in system Sᵢ (i =1, 2, b₁, b₂, h, l, m₁, m₂, p, g)

sᵢ
Surface parameters of the pinion

Sᵢ
Coordinate system (i =1, 2, b₁, b₂, h, l, m₁, m₂, p, g)

Sₜᵦ
Radial setting of the head cutter of the pinion

uᵾ(^k)
Gear tooth surface parameter

vᵾ(k=p, g)
Velocity of contact point in its motion over surface Σₖ

vᵾ(i)
Transfer velocity of contact point in its motion with surface Σᵢ

vᵾ(g)
Relative velocity of contact point (i, j =1, 2, c₁, p)

vᵾ(i), vᵾ(j)
Components of the velocity of the contact point in its motion over surface Σᵢ

Xᵦ, Xₔ
Center distances of fillet circular arcs of pinion and gear, respectively

2ₘ₁
Length of major axis of contact ellipse
1. Introduction

Design and generation of spiral bevel gears is a subject of intensive research of scientists, designers and manufacturers [1 - 7]. High quality equipment such as CNC (computer numerically controlled) machines and tools are produced by the Gleason Works (USA) and the Klingelnber-Orlikon (Germany and Switzerland) that manufacture these gears. However, the quality of spiral bevel gears depends substantially on the choice of machine-tool settings that have to be determined for the generation of the gears. The machine-tool settings are not standardized and their determination is the subject of a skilled design. The freedom of choosing various machine-tool settings provide an opportunity for generation of low-noise, stable bearing contact spiral bevel gears.

The subject of this paper is the enhanced design of face-milled formate cut spiral bevel gears. The term “formate cut” means that the gear is held at rest where it is cut by the tool (head-cutter). Thus, the gear tooth surface is a copy of the tool surface that is a surface of revolution. The pinion tooth surface is generated by a head-cutter while the tool and the pinion performed related rotations. The pinion tooth surface is generated as the envelope to the family of the tool (head-cutter) surfaces. The advantage of formate-cut method is the high productivity of gear generation. The main problem of the design of formate-cut spiral bevel gears is the conjugation of tooth surfaces of the gear and the pinion.

The contents of this paper cover the following topics:
(1) Computational procedures of proposed design that provides a longitudinal bearing contact and a predesigned parabolic function of transmission errors, and the ability to investigate the influence of misalignment.

(2) Computer programs and algorithms developed for synthesis and analysis of meshing and contact of gear drives.

The theory developed is illustrated with an example of computerized design of a formate cut gear drive.

2. Basic Ideas of Proposed Approach

Localization of Contact [1, 8]. The contact of pinion-gear tooth surfaces is localized due to the mismatch of generating surfaces (surfaces of the head-cutters) that is achieved by application of different dimensions of the gear-pinion head-cutters.

The generated pinion-gear tooth surfaces are in point contact at every instant, instead of in line contact. The tooth contact under the load is spread over an elliptical area and the bearing contact is formed as a set of contact ellipses. The magnitude and orientation of instantaneous contact ellipse requires information regarding the elastic deformation of tooth surfaces and the principal curvatures and the directions of tooth surfaces at each instantaneous point of their tangency.

It will be shown later that the path of contact on gear tooth surface is oriented longitudinally. Such a path of contact is favorable for the increase of contact ratio (the average number of teeth that are in contact simultaneously), and avoids edge contact and improves the conditions of transfer of meshing where one pair of teeth is changed for the neighboring one.

The TCA (Tooth Contact Analysis) computer program developed by the authors permits simulation of meshing surfaces and the shift of bearing contact caused by errors of alignment [8, 9]. In some cases of design, it may become necessary to adjust the bearing contact - to deviate it from the longitudinal direction - to reduce the shift of bearing contact caused by misalignment.
Application of Parabolic Profiles of Blades of Gear Head-Cutter. We have mentioned above that the gear tooth surface is generated as a copy of the surface of the gear head-cutter. Usually, straight profile blades are applied for the gear head-cutter. Such blades form the head-cutter generating surface as a cone.

Application of parabolic profiles of the blades instead of straight line profiles permits modification of the gear tooth surface in favor of conjugation of pinion-gear tooth surfaces.

Simulation of Meshing. The algorithm of computerized simulation of meshing [8, 9] provides continuous tangency of contacting gear tooth surfaces $\Sigma_1$ and $\Sigma_2$ as equality of position vectors $r_h^{(1)} = r_h^{(2)}$ and unit normals $n_h^{(1)} = n_h^{(2)}$. Instead of equality of unit normals to contacting surfaces, collinearity of surface normals may be considered. Position vectors $r_h^{(i)} (i=1,2)$ and unit normals $n_h^{(i)} (i=1,2)$ are considered in fixed coordinate system $S_h$ (Fig. 1).

Application of TCA determines the path of contact on the pinion-gear tooth surfaces, dimensions and orientation of instantaneous contact ellipse at each point of contact, the transmission function $\phi_2(\phi_1)$ that relates the angles of rotation of the gear and the pinion, and function $\Delta\phi_2(\phi_1)$ of transmission errors is determined as

$$\Delta\phi_2(\phi_1) = \phi_2(\phi_1) - \frac{N_1}{N_2} \phi_1$$

where $N_1$ and $N_2$ are the numbers of teeth of the pinion and the gear.

Predesign of Parabolic Function of Transmission Errors [8]. The output of TCA for a misaligned gear drive shows that the transmission function $\phi_2(\phi_1)$ is a piecewise discontinuous almost linear function (Fig. 2(a)). The transfer of meshing with the transmission function $\phi_2(\phi_1)$ (Fig. 2(a)) and function $\Delta\phi_2(\phi_1)$ of transmission errors (Fig. 2(b)) is accompanied with the jump of the angular velocity and therefore noise and vibration are inevitable.

The purpose of synthesis of a low-noise gear drive is to transform the transmission function $\phi_2(\phi_1)$ as shown in Fig. 3(a) and obtain a parabolic function of transmission errors shown in Fig. 3(b). The modified function $\Delta\phi_2(\phi_1)$ (Fig. 3(b)) is a continuous one and is determined as

$$\Delta\phi_2^{(i)}(\phi_1) = -a\phi_1^2$$

where $a$ is the parabola coefficient.

It is important to recognize that the function of transmission errors must be negative and therefore the gear will lag with respect to the pinion. Therefore, the transfer of meshing is accompanied with elastic deformations, and as a result, the gear ratio is increased.

The desired parabolic function of transmission errors is provided by the respective synthesis of the gear drive. The idea of such synthesis is based on the predesign of a parabolic function obtained by application of modified roll during the process of generation. Modified roll is provided by application of nonlinear relations between the angles of rotation of the cradle of the generation machine and the pinion during the process of pinion generation.

The advantage of the predesign of a parabolic function of transmission errors is that such a function is able to absorb the linear discontinuous function of transmission errors caused by misalignment. This is
illustrated by drawings of Fig. 4(a) that shows a linear function $\Delta\phi_2^{(1)}(\phi_t) = b\phi_t$ (Fig. 4(a)). Function $\Delta\phi_1^{(1)}(\phi_t)$ is predesigned and function $\Delta\phi_2^{(2)}(\phi_t)$ is caused by misalignment. The sum of functions $\Delta\phi_1^{(1)}(\phi_t)$ and $\Delta\phi_2^{(2)}(\phi_t)$ is represented in Fig. 4(a) as $\Delta\psi_2(\psi_1) = -\alpha\psi_1^2$. The parabola coefficient $a$ is the same in functions $\Delta\phi_2^{(1)}(\phi_t)$ and $\Delta\psi_2(\psi_1)$.

The figure confirms that: (i) the predesigned parabolic function absorbs indeed linear function $\Delta\phi_2^{(2)}(\phi_t)$, and (ii) the resulting function $\Delta\psi_2(\psi_1)$ is slightly dislocated with respect to origin of $\Delta\phi_2^{(1)}(\phi_t)$, function $\Delta\psi_2(\psi_1)$ is an unsymmetrical one. Parameters $c$ and $d$ of the origin of $\Delta\psi_2(\psi_1)$ are determined as $d = b^2/4a$ and $c = b/2a$. In the process of meshing, whereas several cycles of meshing are considered (Fig. 4(b)), the resulting function $\Delta\psi_2(\psi_1)$ of transmission errors becomes a symmetric parabolic function.

The authors have developed computerized synthesis of formate-cut spiral bevel gears that is based on simultaneous application of local synthesis and simulation of meshing and contact of gear tooth surfaces.

The purpose of local synthesis is to determine the pinion machine-tool settings considering as given: (i) the gear machine-tool settings, and (ii) the conditions of meshing and contact at the mean contact point $M$ such as the tangent to the path of contact at $M$, the major axis of the contact ellipse at $M$, and the derivative of the function of transmission errors.

The purpose of simulation of meshing and contact is the determination of the bearing contact and function of transmission errors knowing the pinion and gear machine-tool settings.

The applied computational procedures are iterative processes with the goals to obtain: (i) a longitudinally directed bearing contact, (ii) a parabolic function of transmission errors with limited value of maximal errors, and (iii) reduced sensitivity of the gear drive to errors of alignment.

3. Derivation of Gear Tooth Surface [7, 9]

**Applied Coordinate Systems:** Coordinate system $S_m$ is the fixed one and it is rigidly connected to the cutting machine (Fig. 5). Fig. 5(a) and 5(b) show two sets of coordinate systems applied for generations of left and right hand gears, respectively. Coordinate system $S_2$ is rigidly connected to the gear. Coordinate system $S_g$ is rigidly connected to the gear head-cutter. It is considered that the gear head-cutter is a surface of revolution, and the rotation of the head cutter about the $X_g$– axis does not affect the process of generation.

The installment of the tool on the machine is determined by parameters $H_2$ and $V_2$, that are called horizontal and vertical settings. Parameters $X_G$ and $\gamma_m$ represent the settings of the gear.

**Head Cutter Surface:** The blades of the gear head-cutter of a parabolic profiles are shown in Fig. 6(a). Each side of the blade generates two sub-surfaces. The segment $a$ of the parabolic profile generates the working part of the gear tooth surface. The circular arc of radius $\rho_a$ generates the fillet of the gear tooth surface. The generating surfaces of the head-cutter are formed by rotation of the blade about the $X_g$– axis of the head-cutter (Figs. 6(b) and 6(c)); the rotation angle is $\theta_g$. Therefore, the generating surfaces are: (i) surface of revolution formed by rotation of the blade of parabolic profile, and (ii) the surface of the torus formed by rotation of the circular arc profiles. A point on the generating surface is determined by parameters $u_g$ and $\theta_g$ for the working surface, and by $\lambda_a$ and $\theta_g$ for the fillet surface. Angle $\alpha_g$ is formed
between the tangent line of the blade at point $M$ and the vertical center line of the blade. Parameter $u_g$ measured from point $M$ in the chosen direction is considered as a positive one, and angles $\alpha_g$ and $\lambda_w$ as the acute ones. The apex of the parabola is located at point $M$ determined by parameter $s_{s_{0}}$.

The surface of revolution and the torus surface of the head-cutter are designated as parts $(a)$ and $(b)$ of the head-cutter generating surface. Surface $\Sigma_{g}^{(a)}$ of the head-cutter is represented by vector function $r_{g}^{(a)}(u_g, \theta_g)$ as follows

$$
r_{g}^{(a)}(u_g, \theta_g) = \begin{bmatrix} -u_g \cos \alpha_g + a_u u_g^2 \sin \alpha_g - s_{s_{0}} \cos \alpha_g \\
(R_g \pm s_{s_{0}} \sin \alpha_g \pm u_g \sin \alpha_g \pm a_u u_g \cos \alpha_g) \sin \theta_g \\
(R_g \pm s_{s_{0}} \sin \alpha_g \pm u_g \sin \alpha_g \pm a_u u_g \cos \alpha_g) \cos \theta_g \\
\end{bmatrix}
$$

where $u_g$ and $\theta_g$ are the surface coordinates, $\alpha_g$ is the blade angle at point $M$, $R_g$ is the cutter point radius (Fig. 6) and given by

$$
R_g = R_o \pm \frac{P_{o}}{2}
$$

The upper and lower signs in equations (3) and (4) correspond to the concave and convex sides of the gear tooth, respectively.

The unit normal to the gear generating surface $\Sigma_{g}^{(a)}$ is represented by the equations

$$
n_{g}^{(a)}(u_g, \theta_g) = \frac{N_{g}^{(a)}}{|N_{g}^{(a)}|}, \quad N_{g}^{(a)} = \frac{\partial r_{g}^{(a)}}{\partial u_g} \times \frac{\partial r_{g}^{(a)}}{\partial \theta_g}
$$

Equations (3) and (5) yield

$$
n_{g}^{(a)}(u_g, \theta_g) = \begin{bmatrix} \pm (\sin \alpha_g + 2 a_u u_g \cos \alpha_g) \\
(\cos \alpha_g - 2 a_u u_g \sin \alpha_g) \sin \theta_g \\
(\cos \alpha_g - 2 a_u u_g \sin \alpha_g) \cos \theta_g + \sqrt{1 + 4 a_u^2 u_g^2} \\
\end{bmatrix}
$$

Surface $\Sigma_{g}^{(b)}$ is represented in $S_g$ as

$$
r_{g}^{(b)}(\lambda_w, \theta_g) = \begin{bmatrix} -\rho_w(1 - \cos \lambda_w) \\
(X_g \pm \rho_w \sin \lambda_w) \sin \theta_g \\
(X_g \pm \rho_w \sin \lambda_w) \cos \theta_g \\
\end{bmatrix}
$$

where

$$
X_g = R_g \mp \rho_w(1 - \sin \alpha_g) / \cos \alpha_g
$$

Here, $\rho_w$ is the fillet radius of the gear generating surface.
The unit normal to the gear generating surface \( \Sigma_g^{(b)} \) is represented by the equations

\[
\mathbf{n}_g^{(b)}(\lambda_g, \theta_g) = \mathbf{N}_g^{(b)} \left( \frac{\mathbf{N}_g^{(b)}}{||\mathbf{N}_g^{(b)}||} \right), \quad \mathbf{N}_g^{(b)} = \frac{\partial \mathbf{r}_g^{(b)}}{\partial \lambda_g} \times \frac{\partial \mathbf{r}_g^{(b)}}{\partial \theta_g}
\]  

(8)

Equations (8) yield

\[
\mathbf{n}_g^{(b)}(\lambda_g, \theta_g) = \left[ \begin{array}{c} \pm \cos \lambda_g \\ \sin \lambda_g \sin \theta_g \\ \sin \lambda_g \cos \theta_g \end{array} \right]
\]  

(9)

**Equations of Head-Cutter Surfaces in** \( S_2 \): Applying coordinate transformation from \( S_g \) to \( S_2 \), the gear tooth surfaces can be represented in \( S_2 \) by the equations

\[
\mathbf{r}_2^{(a)}(u_g, \theta_g) = \mathbf{M}_{2g}^{(a)} \mathbf{r}_2^{(a)}(u_g, \theta_g)
\]  

(10)

\[
\mathbf{r}_2^{(b)}(\hat{\lambda}_g, \theta_g) = \mathbf{M}_{2g}^{(b)} \mathbf{r}_2^{(b)}(\hat{\lambda}_g, \theta_g)
\]  

(11)

Here

\[
\mathbf{M}_{2g} = \mathbf{M}_{2m_2} \mathbf{M}_{m_2g}
\]  

(12)

\[
\mathbf{M}_{m_2g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \pm V_2 \\ 0 & 0 & 1 & H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
\mathbf{M}_{2m_2} = \begin{bmatrix} \cos \gamma_{m_2} & 0 & -\sin \gamma_{m_2} & 0 \\ 0 & 1 & 0 & 0 \\ \sin \gamma_{m_2} & 0 & \cos \gamma_{m_2} & -X_G \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

Here: \( V_2 \), \( H_2 \) and \( X_G \) are the gear machine tool settings. The upper and lower signs in front of \( V_2 \) correspond to right hand and left hand gears respectively. The whole set of gear machine tool settings is represented in Table 1. Remember that the generated gear tooth surface is a copy of the surface of the headcutter, that is a surface of revolution. The rotation of the head-cutter about the \( X_g \) – axis that is necessary for the cutting or grinding process does not affect the shape of gear tooth surfaces.

The geometric model of gear tooth surfaces is shown in Fig. 7.

**Table 1: Gear Machine Tool Settings**

<table>
<thead>
<tr>
<th>Blade angle ( \alpha_g ) (Fig. 6)</th>
<th>Blade parabola coefficient ( \alpha_g )</th>
</tr>
</thead>
</table>

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Cutter radius $R_u$ (mm) (Fig. 6)

Point width $P_u$ (mm) (Fig. 6)

Cutter point radius $R_s$ (mm) (Fig. 6)

Horizontal settings $H_2$ mm) (Fig. 5)

Vertical settings $V_2$ (Fig. 5)

Machine center to back $X_c$ (mm) (Fig. 5)

Machine root angle $\gamma_m$ (Fig. 5)

Fillet radius $\rho_m$ (mm) (Fig. 6)

4. Derivation of Pinion Tooth Surface [7, 9, 10]

**Applied Coordinate Systems:** Coordinate systems applied for generation of pinion are shown in Fig. 8 and 9. Fig. 9(a) and (b) correspond to generation of right and left hand pinions, respectively. Coordinate systems $S_{m_1}$, $S_{a_1}$ and $S_{b_1}$ are the fixed ones and they are rigidly connected to the cutting machine. The movable coordinate systems $S_c$ and $S_{c_1}$ are rigidly connected to the pinion and the cradle, respectively. They are rotated about the $Z_{m_1}$ – axis and $Z_{a_1}$ – axis, respectively, and their rotations are related with a polynomial function $\psi_1(\psi_1)$, if modified roll is applied (see below). The ratio of instantaneous angular velocities of the pinion and the cradle is defined as $m_{l_1}(\psi_1(\psi_1)) = \omega^{(1)}(\psi_1)/\omega^{(c)}$. The magnitude of $m_{l_1}(\psi_1)$ at $\psi_{c_1} = 0$ is called ratio of roll. Parameters $X_{D_1}$, $X_{B_1}$, $E_{m_1}$, and $\gamma_{m_1}$ are the basic machine tool settings for pinion generation.

**Head Cutter Surfaces:** The pinion generating surfaces are formed by surface $\Sigma_p^{(a)}$ and $\Sigma_p^{(b)}$ generated by straight-line and circular arc parts of the blades (Fig. 10). Surface $\Sigma_p^{(a)}$ is represented as

$$r_p^{(a)}(s_p, \theta_p) = 
\begin{bmatrix}
  (R_p + s_p \sin \alpha_p) \cos \theta_p \\
  (R_p + s_p \sin \alpha_p) \sin \theta_p \\
  -s_p \cos \alpha_p 
\end{bmatrix}$$

where $s_p$ and $\theta_p$ are the surface coordinates; $\alpha_p$ is the blade angle; $R_p$ is the cutter point radius (Fig. 10).

The upper and lower signs in equations (13) correspond to the convex and concave sides of the pinion tooth respectively.

The unit normal to the pinion generating surface $\Sigma_p^{(a)}$ is represented by the equations

$$n_p^{(a)}(\theta_p) = \frac{n_p^{(a)}(\theta_p)}{|n_p^{(a)}(\theta_p)|}, \quad n_p^{(a)}(\theta_p) = \frac{\partial r_p^{(a)}(\theta_p)}{\partial s_p} \times \frac{\partial r_p^{(a)}(\theta_p)}{\partial \theta_p}$$

Equations (13) and (14) yield

$$n_p^{(a)}(\theta_p) = 
\begin{bmatrix}
  \cos \alpha_p \cos \theta_p \\
  \cos \alpha_p \sin \theta_p \\
  \mp \sin \alpha_p 
\end{bmatrix}$$

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For the fillet surface $\Sigma_p^{(b)}$ we obtain

$$
\mathbf{r}_p^{(b)}(\lambda_f, \theta_p) = 
\begin{bmatrix}
(X_f + \rho_f \sin \lambda_f \cos \theta_p - \rho_f (1 - \cos \lambda_f) \\
(X_f + \rho_f \sin \lambda_f \sin \theta_p) \\
- \rho_f \sin \lambda_f \\
\rho_f (1 - \cos \lambda_f)
\end{bmatrix}
$$

(16)

where

$$
X_f = R_p \pm \rho_f (1 \pm \sin \alpha_p)/\cos \alpha_p
$$

and $\rho_f$ is the radius of the tool fillet.

The unit normal to the pinion generating surface $\Sigma_p^{(b)}$ is represented by the equations

$$
\mathbf{n}_p^{(b)}(\lambda_f, \theta_p) = \frac{\mathbf{N}_p^{(b)}}{\mathbf{N}_p^{(b)}} \quad \mathbf{N}_p^{(b)} = \frac{\partial \mathbf{r}_p^{(b)}}{\partial \lambda_f} \times \frac{\partial \mathbf{r}_p^{(b)}}{\partial \theta_p}
$$

(17)

Equations (17) yield

$$
\mathbf{n}_p^{(b)}(\lambda_f, \theta_p) = 
\begin{bmatrix}
\sin \lambda_f \cos \theta_p \\
\sin \lambda_f \sin \theta_p \\
\mp \cos \lambda_f
\end{bmatrix}
$$

(18)

The head-cutter is mounted on coordinate system $S_i$, called the cradle of the cutting machine and its installment is determined by settings $S_i$ and $q_i$ (Fig. 9). In the process of pinion generation coordinate systems $S_i$ and $S_i$ performed related rotations with respect to $S_m$. Then, in coordinate system $S_i$, a family of pinion head-cutter surfaces is generated and determined as:

$$
\mathbf{r}_i^{(a)}(s, \theta_p, \psi_i) = \mathbf{M}_{ip}(\psi_i) \mathbf{r}_p^{(a)}(s, \theta_p)
$$

(19)

$$
\mathbf{r}_i^{(b)}(\lambda_f, \theta_p, \psi_i) = \mathbf{M}_{ip}(\psi_i) \mathbf{r}_p^{(b)}(\lambda_f, \theta_p)
$$

(20)

where $\psi_i$ is the generalized parameter of motion, and

$$
\mathbf{M}_{ip} = M_{ih} M_{bi} M_{ao} M_{mo} M_{me} M_{ei, p}
$$

$$
\mathbf{M}_{ei, p} = 
\begin{bmatrix}
1 & 0 & 0 & S_i \cos q_i \\
0 & 1 & 0 & \pm S_i \sin q_i \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$
The upper and lower signs of \( S_q \sin q \) in matrix \( \mathbf{M}_{s_p} \) correspond to generation of right and left hand pinions, respectively. Where the modified roll is applied in the process of generation, the rotation angles \( \psi_1 \) and \( \psi_{\epsilon_1} \) of the pinion and cradle are related as

\[
\psi_1 = m_1 \psi_{\epsilon_1} + b_2 \psi_{\epsilon_1}^2 + b_3 \psi_{\epsilon_1}^3 \quad (21)
\]

where \( b_2 \) and \( b_3 \) are the modified roll coefficients.

The derivative of function \( \psi_1(\psi_{\epsilon_1}) \) taken at \( \psi_{\epsilon_1} = 0 \) determines the so-called ratio of roll represented in equation (21) by \( m_1 \).

**Equation of Meshing:** The pinion tooth surface \( \Sigma_1 \) is the envelope to the family of head-cutter surfaces. Surface \( \Sigma_1 \) consists of two surfaces \( \Sigma_1^{(a)} \) and \( \Sigma_1^{(b)} \) which correspond to the cutter surfaces \( \Sigma_p^{(a)} \) and \( \Sigma_p^{(b)} \), respectively. Modified roll is applied in the process of generation. For generation of surface \( \Sigma_1^{(a)} \), the equation of meshing is represented as

\[
\mathbf{n}_{m_1} \cdot \mathbf{v}_{m_1} = s_1^{(a)}(s, \theta, \psi_1) = 0 \quad (22)
\]

where \( \mathbf{n}_{m_1} \) the unit normal to the surface, and \( \mathbf{v}_{m_1} \) is the velocity in relative motion. These vectors are represented in the fixed coordinate system \( S_m \) as follows:

\[
\mathbf{n}_{m_1}^{(a)} = \mathbf{L}_{m_1} \mathbf{L}_{c_1} \mathbf{n}_{p}^{(a)}(\theta_p) \quad (23)
\]
\[
\mathbf{v}_{m_i}^{(p)} = [(\mathbf{w}_{m_i}^{(p)} - \mathbf{w}_{m_i}^{(1)}) \times \mathbf{r}_{m_i} - (\overrightarrow{O_{m_i}O_{a_i}} \times \mathbf{w}_{m_i}^{(1)})]
\] (24)

Here

\[
\mathbf{r}_{m_i} = M_{m_i}v_m, M_{m_p}r_p^{(a)}(s_p, \theta_p)
\]

\[
\overrightarrow{O_{m_i}O_{a_i}} = \left[0, -E_{m_i}, X_{B_i}\right]^T
\]

\[
\mathbf{w}_{m_i}^{(1)} = \begin{bmatrix} \cos \gamma_{m_i}, 0, \sin \gamma_{m_i} \end{bmatrix}^T
\]

\[
\mathbf{w}_{m_i}^{(p)} = \begin{bmatrix} 0, 0, m_{1,1}(\psi_i) \end{bmatrix}^T
\]

The ratio \(m_{1,1}\) is not constant since modified roll is applied and can be represented as

\[
m_{1,1} = \frac{\omega_{s_i}}{\omega_i} = \frac{d\psi_{s_i}}{dt} / \frac{d\psi_i}{dt} = \frac{1}{m_u + 2b_2\psi_i + 3b_3\psi_i^2}
\] (25)

where \(b_2\) and \(b_3\) are modified roll coefficients.

Finally, we obtained the equations for pinion tooth surface part (a) as

\[
\mathbf{r}_p^{(a)}(s_p, \theta_p, \psi_i) = M_{p,1}(\psi_i)\mathbf{r}_p^{(a)}(s_p, \theta_p)
\] (26)

\[
f_p^{(a)}(s_p, \theta_p, \psi_i) = 0
\] (27)

Similarly, the fillet surface of pinion tooth surface \(\Sigma_i^{(b)}\) may be represented as

\[
\mathbf{r}_p^{(b)}(\lambda_f, \theta_p, \psi_i) = M_{p,1}(\psi_i)\mathbf{r}_p^{(b)}(\lambda_f, \theta_p)
\] (28)

\[
f_p^{(b)}(\lambda_f, \theta_p, \psi_i) = 0
\] (29)

The geometric model of pinion tooth surfaces is shown in Fig. 11.

5. Procedures of Proposed Design

The authors have developed an approach for the design of formate cut spiral bevel gears that provides a stabilized bearing contact and reduced level of noise. The bearing contact is of a longitudinal direction that enables to increase the contact ratio, avoid edge contact and reduce the shift of the bearing contact caused by misalignment. The reduction of noise is achieved by application of a predesigned parabolic function of transmission errors and limitation of maximal transmission errors caused by misalignment. The approach is based on three procedures that require simultaneous execution of computer programs developed for the local synthesis and simulation of meshing and contact of the gear drive.

The purpose of the local synthesis is to determine the pinion machine-tool settings for the following conditions of meshing and contact of pinion-gear tooth surfaces \(\Sigma_i\) and \(\Sigma_2\):

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(i) The gear machine-tool settings are considered as given (they may be adapted, for instance, from a
Gleason summary for gear generation);

(ii) The pinion-gear tooth surfaces $\Sigma_1$ and $\Sigma_2$ are in tangency at a chosen mean contact point $M$ (Fig. 12);

(iii) Parameters $2a$, $\eta_2$, and $m_{12}$ are assigned ahead and determine the conditions of meshing and contact of
$\Sigma_1$ and $\Sigma_2$ at point $M$ and in the neighborhood of $M$. Here, $2a$ is the major axis of the instantaneous contact
ellipse, $\eta_2$ determines the tangent to the path of contact on the gear tooth surface at $M$, and $m_{12}$ is the
derivative \( \frac{\partial}{\partial \phi_1} (m_{12}(\phi_1)) \) at point $M$, where $m_{12}(\phi_1)$ is the gear ratio.

Note: It will be shown in section 6 that the location of point $M_2$ on the gear tooth surface as the candidate
for the mean contact point of $\Sigma_1$ and $\Sigma_2$ may be chosen. Then, the procedure of the local synthesis permits
the determination of point $M_1$ on pinion tooth surface $\Sigma_1$ that will be in tangency with $\Sigma_2$ at the chosen
point $M_2$ of $\Sigma_2$.

The simulation of meshing and contact of pinion-gear tooth surfaces is accomplished by application of the
TCA (Tooth Contact Analysis) computer program developed. The input for TCA are the equations of
pinion-gear tooth surfaces, parameters of motion and assembly. The output is the bearing contact and the
function of transmission errors.

The design is based on simultaneous execution of computer programs developed for local synthesis and
TCA, it is an iterative process and requires application of following three procedures.

**Procedure 1:**

The path of contact on gear tooth surface is a spatial curve $L$. Projection of $L$ on tangent plane $T$ (Fig. 13) is
designated by $L_T$. The purpose of procedure 1 is to obtain that $L_T$ is a straight line directed longitudinally.

Fig. 13 shows projections $L_T^{(1)}$ and $L_T^{(2)}$ that deviate from the desired line $L_T$.

Fig. 14 shows the flow chart that describes the steps of procedure 1. The procedure of computation is an
iterative process and requires variation of parameter $m_{12}$. The steps of the procedure are as follows:

Step 1: Input the gear machine-tool settings into the program of local synthesis.

Step 2: Input parameters $2a$, $\eta_2$, and $m_{12}$ into the program of local synthesis.

Step 3: Execute the program of local synthesis.

Step 4: Determine from the program of local synthesis the pinion machine-tool settings.

Step 5: Using the machine-tool settings obtained, compute the pinion and gear tooth surfaces $\Sigma_1$ and $\Sigma_2$.

Step 6: Execute TCA computer program and obtain projection $L_T$ of path of contact $L$ on tangent plane $T$
(Fig. 13).

Step 7: Execute the iterative process and obtain projection of path of contact $L_T$ as a straight line by
variation of $m_{12}$.

Step 8: Output the final machine-tool settings for the pinion.

The iterative process is based on the following considerations:
(1) The TCA computer program applied determines projection $L_T$ of path of contact numerically.
(2) Consider $L_T$ in coordinate system $S_T$ (Fig. 13) which origin coincides with $M$ and axes $(x_T, y_T)$ are
located in plane $T$. Axis $x_T$ is the tangent to $L_T$ at point $M$.  

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(3) Represent projection $L_r$ as a parabolic regression equation by using a subroutine of regression [11]

$$y_i(x_i, m_{i2}) = \beta_0(m_{i2}) + \beta_1(m_{i2})x_i + \beta_2(m_{i2})x_i^2$$  \hspace{1cm} (30)

where $m_{i2}$ is the input parameter that is varied in the process of iteration.

(4) The goal is to represent $L_r$ as a straight line that might be obtained by such a magnitude of $m_{i2}$ whereas coefficient $\beta_2(m_{i2}) = 0$ (see equation (30)). The iterations for determination of $\beta_2(m_{i2}) = 0$ are executed by applying the secant method [12] and illustrated by Fig. 15. Equation (30) with $\beta_2(m_{i2}) = 0$ provides the projection of path of contact as a straight line.

**Procedure 2:**

Procedure 1 discussed above provides the desired longitudinal path of contact and bearing contact. However, the shape of the obtained function of transmission errors and the magnitude of maximal transmission errors do not satisfy the requirements of low-noise gear drive. The goal of procedure 2 is to obtain a parabolic function of negative transmission errors and of limited value of maximal errors.

The sequence of steps applied for procedure 2 is represented by the flow chart shown in Fig. 16.

At the start, we use the pinion and gear machine tool settings obtained in procedure 1, but consider that the pinion is generated by application of modified roll. Then, we compute the gear and pinion tooth surfaces $\Sigma_1$ and $\Sigma_2$, and use them for the TCA computer program. The intermediate output of TCA is a function of transmission errors. We compare it with the sought-for parabolic function of transmission errors and determine the necessary corrections using modified roll. Repeating the procedure discussed above, we can obtain the sought-for parabolic function of transmission errors.

Analytically, the algorithm of procedure 2 is represented as follows.

**Step 1:** Function of transmission errors $\Delta\phi_2(\phi_i)$ obtained from TCA is numerically represented as a polynomial function up to third order to be included [11]

$$\Delta\phi_2(\phi_i) = a_0 + a_1\phi_i + a_2\phi_i^2 + a_3\phi_i^3$$  \hspace{1cm} (31)

The shape and magnitude of maximal errors of function $\Delta\phi_2(\phi_i)$ do not satisfy the requirements of a low-noise gear drive. Function (31) has to be transformed into a parabolic function with limited magnitude of transmission errors by application of procedure 2.

**Step 2:** We apply for transformation of function $\Delta\phi_2(\phi_i)$ the modified roll for generation of the pinion that is represented as

$$\psi_i(\psi_{e_i}) = m_{e_i}\psi_{e_i} + b_1\psi_{e_i}^2 + b_2\psi_{e_i}^3$$  \hspace{1cm} (32)

Here, $\psi_i$ and $\psi_e$ are the angles of rotation of the pinion and the cradle performed during the process for generation, $m_{e_i}$ is the first derivative of $\psi_i(\psi_{e_i})$ taken at $\psi_{e_i} = 0$ and obtained by the procedure of local
synthesis (see section 6). Coefficients \( b_2 \) and \( b_3 \) have to be determined by using an iterative process (see below).

Step 3: Assigning coefficients \( b_2 \) and \( b_3 \) of modified roll and using pinion machine-tool settings obtained in procedure 1, we may determine the pinion tooth surface \( \Sigma_1 \). The gear tooth surface \( \Sigma_2 \) is considered as given.

Step 4: Applying TCA for simulation of meshing of \( \Sigma_1 \) and \( \Sigma_2 \) we may obtain numerically the modified function of transmission errors and then represent it again by polynomial function (31) with new magnitudes of coefficients \( a_0, a_1, a_2 \) and \( a_3 \).

Procedure 2 with steps 1 to 4 has to be repeated until a parabolic function of transmission errors \( \Delta\phi_2(\phi_1) \) with limited magnitude of maximal transmission errors is obtained.

The computation is an iterative process based on application of secant method of numerical computations [12]. The goal is to transform the shape of obtained function \( \Delta\phi_2^{(i)}(\phi_1) \) of transmission errors into a parabolic function

\[
\Delta\phi_2(\phi_1) = -a_3\phi_1^2, \quad -\frac{\pi}{N_1} \leq \phi_1 \leq \frac{\pi}{N_1}
\]

(33)

where

\[
\left| \Delta\phi_2(\phi_1) \right|_{max} = a_3\left( \frac{\pi}{N_1} \right)^2 = \Delta\Phi
\]

(34)

This goal is accomplished by variation of coefficients \( b_2 \) and \( b_3 \) of function (32) provided by modified roll. The variation of \( b_2 \) and \( b_3 \) is performed independently and is illustrated by drawings of Fig. 17.

Fig. 17(a) illustrates variation of coefficient \( b_3 \) of modified roll for obtaining of coefficient \( a_3 = 0 \) of function (31). Function \( a_3(b_2) \) is determined as the output of TCA by variation of modified roll.

Fig. 17(b) illustrates variation of coefficient \( b_2 \) of modified roll for obtaining of \( \left| \Delta\phi_2(\phi_1) \right|_{max} = \Delta\Phi \).

Function \( \Delta\phi_2(\phi_1) \) is determined as the output of TCA by variation of \( b_2 \).

Procedure 3:

Procedures 1 and 2 are performed by simultaneous application of computer programs of local synthesis and TCA, but the errors of alignment of the gear drive are not considered.

The purpose of procedure 3 is to adjust the gear drive for the existence of errors of alignment. This is achieved by adjusting the previously designed path of contact to the assigned or expected errors of alignment. The procedure is based on the flow chart shown in Fig. 18.

The adjustment of the gear drive to the errors of alignment is performed by correction of parameter \( \eta_z \) (Fig. 12) that determines the orientation of the path of contact on the gear tooth surface.
The prescribed procedure yields that in some cases of design it becomes necessary to deviate the path of contact from the longitudinal direction to reduce the shift of bearing contact by errors of alignment. The procedure has to be applied for separate simulation of each error of alignment for obtaining the required correction of \( \eta_2 \). The permissible tolerance of each alignment error is obtained by procedure 3.

**Local Synthesis and Determination of Pinion Machine Tool Settings** [7, 8, 9]

Remembering that the local synthesis is accomplished simultaneously with the TCA, and the computation is an iterative process. The computer program developed permits the determination at each iteration the pinion machine-tool settings.

The input parameters for the local synthesis are the gear machine-tool settings and parameters \( \eta_2 \), \( a \) and \( m_{12} \) (see Flow chart, Fig. 14).

The procedure of the local synthesis is as follows.

**Stage 1:** Determination of the mean contact point on the gear tooth surface

**Step 1:** Mean point A on surface \( \Sigma_2 \) is chosen by designation of parameters \( L_A \) and \( R_A \) (Fig. 19), where \( A \) is the candidate for the mean contact point \( M \) of surfaces \( \Sigma_2 \) and \( \Sigma_1 \). Then we obtain the following system of two equations in two unknowns

\[
\begin{align*}
Z_2(u^*_g, \theta^*_g) &= L_A \\
X_2^2(u^*_g, \theta^*_g) + Y_2^2(u^*_g, \theta^*_g) &= R_A^2
\end{align*}
\]  

(35)

where \( X_2 \), \( Y_2 \) and \( Z_2 \) are the projections of position vector \( r_g(u^*_g, \theta^*_g) \).

**Step 2:** Equations (35) considered simultaneously permits the surface parameters \((u^*_g, \theta^*_g)\) to be determined for point \( A \). Vector functions \( r_g(u^*_g, \theta^*_g) \) and \( n_g(\theta^*_g) \) determine the position vector and surface unit normal for a current point of gear tooth surface \( \Sigma_g \). Taking in these vector functions \( u_g = u^*_g \) and \( \theta_g = \theta^*_g \), we can determine the position vector \( r_g^{(A)} \) of point \( A \) and the surface unit normal \( n_g^{(A)} \) at \( A \).

**Step 3:** Parameters \( u^*_g \) and \( \theta^*_g \) and the unit vectors \( e_g \) and \( e_u \) of principal directions on surface \( \Sigma_g \) are considered as known. Here:

\[
e_g = \frac{\partial r_g^{(A)}}{\partial u_g} + \frac{\partial r_g^{(A)}}{\partial \theta_g} = \left[ \begin{array}{c}
-\cos \alpha_g + 2a_u u_g \sin \alpha_g \\
(\pm \sin \alpha_g \pm 2a_u u_g \cos \alpha_g) \sin \theta_g \\
(\pm \sin \alpha_g \pm 2a_u u_g \cos \alpha_g) \cos \theta_g \\
\sqrt{1 + 4a^2_u u^2_g}
\end{array} \right]
\]  

(36)
\[ e_u = \frac{\partial x_s^{(a)}}{\partial \theta_g} + \frac{\partial y_s^{(a)}}{\partial \theta_g} = \begin{bmatrix} 0 \\ \cos \theta_g \\ -\sin \theta_g \end{bmatrix} \]  

(37)

The gear generating surface is a surface of revolution, and the principal curvatures \( k_g \) and \( k_u \) of \( \Sigma_g \) are determined by the following equations:

\[
\begin{align*}
  k_g &= \frac{2a_r}{(1 + 4a_r^2 u_s^2) ^ {3/2}} \\
  k_u &= \frac{-\cos \alpha_s + 2a_r u_s \sin \alpha_s}{(R_s \pm s_r \sin \alpha_g \pm u_s \sin \alpha_s \pm a_r \cos \alpha_s)(1 + 4a_r^2 u_s^2)^{1/2}} 
\end{align*}
\]

(38)

The upper and lower signs correspond to the concave and convex sides of gear tooth, respectively.

The unit vectors \( e_i \) and \( e_q \) of principal directions on surface \( \Sigma_2 \) are obtained by using matrix transformation from \( S_g \) to \( S_2 \). The gear principal curvatures \( k_g \) and \( k_u \) of \( \Sigma_2 \) are the same as the principal curvatures \( k_g \) and \( k_u \), respectively, of the gear head-cutter.

**Stage 2:** Tangency of Surfaces \( \Sigma_2 \) (\( \Sigma_g \)) and \( \Sigma_i \) at Mean Contact Point \( M \).

**Step 1:** The derivations accomplished at stage 1 permits the position vector \( r_2^{(A)} \) to be determined and the surface unit normal \( n_2^{(A)} \) of point \( A \) of tangency of surfaces \( \Sigma_2 \) (\( \Sigma_g \)). The goal now is to determine such a point \( M \) in the fixed coordinate system \( S_l \) (Fig. 20) where two surfaces \( \Sigma_2 \) (\( \Sigma_g \)) and \( \Sigma_i \) are in tangency with each other.

Surfaces \( \Sigma_2 \) and \( \Sigma_g \) are the same ones, because there is no relative motion between the cradle and workpiece (between coordinate system \( S_2 \) to \( S_g \) whereas the gear is generated. Using coordinate transformation from \( S_2 \) to \( S_l \) (Fig. 20), we may obtain \( r_l^{(A)} \) and \( n_l^{(A)} \). The new position of point \( A \) in \( S_l \) will become point \( M \) of tangency of \( \Sigma_2 \) and \( \Sigma_i \), if the following equation of meshing between \( \Sigma_2 \) and \( \Sigma_i \) is observed

\[ n_l^{(A)} \cdot (\phi_2^{(0)}) \cdot v_l^{(21,A)} (\phi_2^{(0)}) = 0 \]

(39)

Here, \( n_l^{(A)} \equiv n_l^{(M)} \) and \( v_l^{(12,A)} \equiv v_l^{(12,M)} \); \( v_l^{(12,A)} \) is the relative velocity at point \( A \) determined with the ideal gear ratio

\[ m_{12}^{(0)} = \frac{\omega^{(2)}}{\omega^{(3)}} \]

(40)

The solution of equation (39) for \( \phi_2^{(0)} \) provides the value of the initial turning angle for \( \phi_2 \). It is evident that two surfaces \( \Sigma_2 \) and \( \Sigma_i \) are now in tangency with each other at point \( M \).
Step 2: We consider as known at point $M$ the principal curvatures $k_y$ and $k_q$ of surface $\Sigma_2$, and the unit vectors $e_y$ and $e_q$ of principal directions on $\Sigma_2$. The unit vectors $e_y$ and $e_q$ are represented in $S_2$. The goal now is to determine at point $M$ the principal curvatures $k_f$ and $k_h$ of surface $\Sigma_1$, and the unit vectors $f_y$ and $h_y$ of principal directions on $\Sigma_1$. This goal can be achieved by application of the procedure described in Appendix 1. It is shown in Appendix 1 that the determination of $k_f$, $k_h$, $f_y$ and $h_y$ becomes possible if parameters $m_{1z}$, $\eta_2$ (or $\eta_1$), and $a/\delta$ are assumed to be known or are used as input data.

Stage 3: Tangency of Surfaces $\Sigma_2$, $\Sigma_1$ and $\Sigma_p$ at Mean Contact Point $M$.

Basic Equations: Tangency of $\Sigma_2$ and $\Sigma_1$ at mean contact point $M$ has been already provided at the previous stages. The position vector $r^{(M)}_m$ of point $M$ and the surface unit normal $n^{(M)}_m$ at point $M$ were determined in coordinate system $S_2$. Let us imagine now that coordinate system $S_2$ that coincides with $S_1$ and surface $\Sigma_1$ are installed in coordinate system $S_m$ (Fig. 8). Let angle $\psi^{(0)}_r$ be the installment angle of the pinion (it is the initial value of $\psi_r$). Using coordinate transformation from $S_1$ to $S_m$, we may determine in $S_m$ position vector $r^{(M)}_m$ of point $M$ and the surface unit normal $n^{(M)}_m$. In Appendix 1 the conditions of improved meshing and contact of pinion and gear tooth surfaces $\Sigma_1$ and $\Sigma_2$ are considered and then the relationships between the principal curvatures and directions of surfaces for such conditions of meshing and contact are determined (see equations (93)). The point of tangency of surfaces $\Sigma_1$ and $\Sigma_p$ is designated in Appendix 1 as point $B$. The pinion generating surface $\Sigma_p$ is installed in $S_m$ taking the cradle angle $\psi_{cr}$ equal to zero. The position vector of point $B$ of surface $\Sigma_p$ and the surface unit normal at $B$ are represented in $S_m$ as $r^{(B)}_m$ and $n^{(B)}_m$. The tangency of $\Sigma_1$ and $\Sigma_p$ at the mean contact point $M$ is satisfied, if the following vector equations are observed

$$n^{(M)}_m = n^{(B)}_m$$  \hspace{1cm} (41)

$$r^{(M)}_m = r^{(B)}_m$$  \hspace{1cm} (42)

$$n^{(M)}_m \cdot v^{(p)}_m = 0$$  \hspace{1cm} (43)

where equation (43) is the equation of meshing. Observation of equations (41)-(43) means that all of the three surfaces ($\Sigma_2$, $\Sigma_1$, and $\Sigma_p$) are in contact at point $M$.

Using equations (41)-(43) and (93) from Appendix 1, it becomes possible to obtain the settings of the pinion and the head-cutter that guarantee the improved conditions of meshing and contact at point $M$. The machine tool settings to be determined are as follows:

(i) $X_B$, $E_m$, $X_{D_t}$ and $m_{ip} = \omega^{(1)}/\omega^{(p)}$. Settings $X_B$ and $X_{D_t}$ are related by the following equation (Fig. 21)

$$X_B = -(X_{D_t} + [O_{D_t} - O_{D_1}]\sin \gamma_m)\sin \gamma_m$$  \hspace{1cm} (44)

where

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\[
\frac{\mathbf{O}_p \mathbf{O}_r}{\tan \gamma_{m_r}} = \frac{[(A_w + F_w/2) \sin \gamma_i - a_{\gamma} \cos \gamma_i]}{[-((A_w + F_w/2) \cos \gamma_i - a_{\gamma} \sin \gamma_i)]}
\]

Equation (44) is derived for the case of design of a spiral bevel gear with different apexes of the pitch and root cones.

(ii) Design parameter \( R_p \) of the head-cutter (Fig. 10).

(iii) Parameters \( S_i \) and \( q_i \) that determine the installment of the head-cutter on the cradle (Fig. 9).

(iv) Parameter \( \psi_i^{(0)} \) is the initial angle \( \psi_i \) for installment of coordinate system \( S_i \) with respect to \( S_b \) (Fig. 8), and parameter \( \theta_p \) of the head-cutter surface \( \Sigma_p \).

**Computation of Pinion Machine-Tool Settings.** The procedure for computation is as follows:

**Step 1:** Determine the values of \( \theta_p \) and \( \psi_i^{(0)} \) that are the sought-for two unknowns. Equation (41) is used for determination of \( \theta_p \) and \( \psi_i^{(0)} \), taking into account that

\[
\mathbf{n}_{m_i}^{(M)}(\theta_p) = \begin{bmatrix} \cos \alpha_p \cos \theta_p \\ \cos \alpha_p \sin \theta_p \\ \mp \sin \alpha_p \end{bmatrix}
\]

and

\[
\mathbf{n}_{m_i}^{(M)} = \mathbf{L}_{m_i b} \mathbf{L}_{b i} \mathbf{n}_i^{(M)}
\]

where \( \mathbf{n}_i^{(M)} = \mathbf{n}_i^{(M)} \) since \( S_i \) coincides with \( S_e \) (Fig. 20). Here,

\[
\mathbf{L}_{m_i b} = \begin{bmatrix} \sin \gamma_{m_i} & 0 & \cos \gamma_{m_i} \\ 0 & 1 & 0 \\ -\cos \gamma_{m_i} & 0 & \sin \gamma_{m_i} \end{bmatrix}
\]

\[
\mathbf{L}_{b i} = \begin{bmatrix} \cos \psi_i^{(0)} & -\sin \psi_i^{(0)} & 0 \\ \sin \psi_i^{(0)} & \cos \psi_i^{(0)} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Equations (41) and (45)-(48) yield the following expressions for determination of \( \theta_p \) and \( \psi_i^{(0)} \)

\[
\cos \theta_p = \frac{n_i \pm \sin \gamma_{m_i} \sin \alpha_p}{\cos \gamma_{m_i} \cos \alpha_p}
\]

\[
\sin \psi_i^{(0)} = \frac{-n_i \cos \theta_p \cos \alpha_p + n_i \sin \gamma_{m_i} \sin \theta_p \cos \alpha_p + n_i n_l \cos \gamma_{m_i}}{(n_i + n_l) \sin \gamma_{m_i}}
\]
\[
\cos \psi_1^{(B)} = \frac{\sin \theta_p \cos \alpha_p - n_h \sin \psi_1^{(B)}}{n_b}
\]

where \( \alpha_p \) is the given value of the profile angle of the head-cutter, and \( n_h \), \( n_b \), and \( n_e \) are the three components of vector \( \mathbf{n}_b^{(M)} \). The great advantage of the approach developed is that the requirement of the coincidence of the normals does not require a non-standard profile angle \( \alpha_p \) or the tilt of the head-cutter with respect to the cradle.

Using \( \theta_p \), it becomes possible to determine the unit vectors \( \mathbf{e}_{\psi} \) and \( \mathbf{e}_{e} \) of principal directions on surface \( \Sigma_p \) at point \( M \).

**Step 2:** Determination of pinion machine-tool settings \( X_{B_i} \), \( X_{D_i} \), \( E_{m_i} \), \( m_{t_p} \), and the design parameter \( R_p \) of the head-cutter (five unknowns are sought-for).

As a reminder, \( X_{B_i} \) and \( X_{D_i} \) are related by equation (44). The determination of the machine-tool settings mentioned above is based on application of the system of equations (93) and equation (43) that represent a system of four non-linear equations with four unknowns: \( X_{D_i} \), \( E_{m_i} \), \( m_{t_p} \), and \( R_p \).

The design parameters mentioned above provide as well improved conditions of meshing and contact at the mean contact point \( M \).

**Step 3:** Determination of machine-tool settings \( r_{S_i} \) and \( q_{i} \) and the pinion surface parameter \( s_p \) (three unknowns are sought-for).

Determination of the mentioned three parameters is based on application of equation (42), considering that generating surface \( \Sigma_p \) is a cone. The final equations for a right hand pinion are as follows:

\[
S_i \cos q_{i} + (R_p \mp s_p \sin \alpha_p) \cos \theta_p = X_{m1}^{(M)}
\]

\[
S_i \sin(\pm q_{i}) + (R_p \mp s_p \sin \alpha_p) \sin \theta_p = Y_{m1}^{(M)}
\]

\[-s_p \cos \alpha_p = Z_{m1}^{(M)}
\]

The upper and lower signs in front of \( q_{i} \) in equation (53) correspond to the design of right-hand and left hand pinions, respectively.

The stages of computation discussed above may be summarized as follows:

(i) It is necessary to determine ten unknowns: six machine-tool settings \( X_{B_i} \), \( E_{m_i} \), \( X_{D_i} \), \( q_{i} \), \( S_i \), \( m_{t_p} \), two surface parameters \( \theta_p \), \( s_p \), one cutter parameter \( R_p \), and one position parameter \( \psi_1^{(B)} \) which defines the pinion initial turn angle.

(ii) The equation system for determination of the unknowns is formed as follows:

\[
\mathbf{n}_{m_i}^{(M)} = \mathbf{n}_{m_i}^{(B)}
\]
\[ \mathbf{r}_{m_1}^{(M)} = \mathbf{r}_{m_1}^{(B)} \quad (56) \]
\[ \mathbf{n}_{m_1}^{(M)} \cdot \mathbf{v}_{m_1}^{(1p)} = 0 \quad (57) \]
\[ X_{n_i} = -(X_{n_i} - |O_{n_i}O|) \sin \gamma_{m_i} \quad (58) \]

In addition, we use three curvature equations

\[
\begin{align*}
\tan 2\sigma_{(1p)}^{(1p)} &= \frac{-2d_1d_{23}}{d_{33}^2 - d_{13}^2 - (k_f - k_h)d_{33}} \\
k_i - k_p &= \frac{-2d_1d_{23}}{d_{33} \sin 2\sigma_{(1p)}^{(1p)}} \\
k_i + k_p &= k_f + k_h + \frac{d_{13}^2 + d_{23}^2}{d_{33}}
\end{align*}
\quad (59)
\]

Equation (55) is equivalent to two independent scalar equations, equations (56) is equivalent to three scalar equations, equations (57), (58) and (59) represent five scalar equations. Thus, the system of equations provides indeed ten scalar equations for determination of ten unknowns. The solution for the unknowns requires solution of a subsystem of four nonlinear equations (see step 2) and solution of six remaining equations represented in echelon form (each of the six equations contains one unknown to be determined).

### 7. Simulation of Meshing and Contact Accomplished in Procedure 1 (Section 5) [7, 8, 9]

The main goal of simulation of meshing and contact in procedure 1 (see Flow chart of Fig. 14) is to determine bearing contact that corresponds to pinion machine-tool settings obtained in section 6. Remembering that the computational procedures represented in sections 6 and 7 must be applied simultaneously, to represent an iterative process for determination of longitudinal path of contact. The developed TCA (Tooth Contact Analysis) computer program (based on simulation of meshing and contact) provides at each stage of iteration the path of contact. In addition, TCA provides the function of transmission errors which shape has to be corrected by application of pinion modified roll (see procedure 2 of section 5).

**Applied Coordinate Systems:** The meshing of gear tooth surfaces is considered in the fixed coordinate system \( S_h \) that is rigidly connected to the housing (Fig. 22). Movable coordinate system \( S_1 \) and \( S_2 \) are rigidly connected to the pinion and the gear, respectively. Auxiliary coordinate systems \( S_{h_1} \) and \( S_{h_2} \) are used to represent the rotation of the pinion (with respect to \( S_{h_1} \)) and the gear (with respect to \( S_{h_2} \)). The errors of alignment are simulated by respective installment of \( S_{h_1} \) and \( S_{h_2} \) with respect to \( S_h \).

The errors of installment are: \( \Delta H \) – the axial displacement of the pinion; \( \Delta \gamma \) – the change of the shaft angle \( \gamma \); \( \Delta E \) – the shortest distance between the axes of the pinion and the gear when the pinion-gear axes are crossed instead of intersected; \( \Delta D \) – the axial displacement of the gear. In the case of an aligned gear drive we consider that \( \Delta H \), \( \Delta \gamma \), \( \Delta E \) and \( \Delta D \) are all equal to zero.
Simulation Algorithm: During the process of meshing, the pinion and gear tooth surfaces must be in continuous tangency and this can be provided, if at any instant their position vectors coincide and the surface normals are collinear. Instead of collinearity of surface normals, equality of surface unit normals may be required.

Pinion and gear tooth surfaces are represented in coordinate system $S_h$ by the following equations

$$
\mathbf{r}_h^{(1)}(s, \theta, \psi, \phi) = \mathbf{M}_{h,1} \mathbf{r}_1(s, \theta, \psi, \phi)
$$

$$
\mathbf{r}_h^{(2)}(u, \theta, \phi) = \mathbf{M}_{h,2} \mathbf{r}_2(u, \theta, \phi)
$$

Here:

$$
\mathbf{M}_{h,1} = \begin{bmatrix}
\cos \phi_1 & \sin \phi_1 & 0 & 0 \\
-\sin \phi_1 & \cos \phi_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\mathbf{M}_{h,2} = \begin{bmatrix}
\cos \phi_2 & -\sin \phi_2 & 0 & 0 \\
\sin \phi_2 & \cos \phi_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\mathbf{M}_{h,3} = \begin{bmatrix}
\cos(\gamma + \Delta \gamma) & 0 & -\sin(\gamma + \Delta \gamma) & -\Delta D \sin(\gamma + \Delta \gamma) \\
0 & 1 & 0 & \Delta E \\
\sin(\gamma + \Delta \gamma) & 0 & \cos(\gamma + \Delta \gamma) & \Delta D \cos(\gamma + \Delta \gamma) \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Unit normals to surfaces of pinion and gear are represented in $S_h$ by the following equations, respectively

$$
\mathbf{n}_h^{(1)}(s, \theta, \psi, \phi) = \mathbf{L}_{h,1}(\phi) \mathbf{n}_1(s, \theta, \psi, \phi)
$$

$$
\mathbf{n}_h^{(2)}(u, \theta, \phi) = \mathbf{L}_{h,2}(\phi) \mathbf{n}_2(u, \theta, \phi)
$$

Here:

$$
\mathbf{L}_{h,1} = \begin{bmatrix}
\cos \phi_1 & \sin \phi_1 & 0 \\
-\sin \phi_1 & \cos \phi_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$
\[ \mathbf{L}_{b_{i2}} = \begin{bmatrix} \cos \phi_2 & -\sin \phi_2 & 0 \\ \sin \phi_2 & \cos \phi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \mathbf{L}_{hb_i} = \begin{bmatrix} \cos(\gamma + \Delta \gamma) & 0 & -\sin(\gamma + \Delta \gamma) \\ 0 & 1 & 0 \\ \sin(\gamma + \Delta \gamma) & 0 & \cos(\gamma + \Delta \gamma) \end{bmatrix} \]

Conditions of continuous tangency of pinion and gear tooth surfaces are represented by the following equations

\[ r_n^{(1)}(s_p, \theta_p, \psi_1, \phi_1) - r_n^{(2)}(u_g, \theta_g, \phi_2) = 0 \quad (64) \]

\[ n_n^{(1)}(s_p, \theta_p, \psi_1, \phi_1) - n_n^{(2)}(u_g, \theta_g, \phi_2) = 0 \quad (65) \]

\[ f_{1p}^{(a)}(s_p, \theta_p, \psi_1) = 0 \quad (66) \]

Surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are represented in \( S_0 \) by four and three related parameters, respectively. Equation (66) is the equation of meshing of the pinion and generating head-cutter and relates the surface parameters of the head-cutter with the generalized parameter \( \psi_1 \) of motion of the process of generation of the pinion. Vector equations (64), (65) and (66) yield three, two independent and one scalar equations, respectively. The whole system of equations (64) – (66) provides six equations for determination of seven unknowns represented as

\[ f_i(s_p, \theta_p, \psi_1, \phi_1, u_g, \theta_g, \phi_2) = 0, \quad f_i \in C^1, \quad (i = 1, ..., 6) \quad (67) \]

One of the unknowns, say \( \phi_1 \), is considered as the input parameter in the range \(-\frac{\pi}{N_1} \leq \phi_1 \leq \frac{\pi}{N_1}\). It is assumed that the Jacobean of the system (67) differs from zero at each iteration.

The paths of contact on the pinion and the gear tooth surfaces are represented by the following functions:

\[ \mathbf{r}_1 = \mathbf{r}_1(s_p, \theta_p, \psi_1, \phi_1) \quad (68) \]

\[ f_{1p}^{(a)}(s_p, \phi_1, \theta_p, \psi_1, \phi_1) = 0 \quad (69) \]

and

\[ \mathbf{r}_2 = \mathbf{r}_2(u_g, \phi_1, \theta_g, \phi_1) \quad (70) \]

respectively.

The function of transmission errors is defined as

\[ \Delta \phi_2(\phi_1) = \phi_2(\phi_1) - \frac{N_1}{N_2} \phi_1 \quad (71) \]
The bearing contact is formed as a set of instantaneous contact ellipses. The lengths of the major and minor axes of contact ellipse and their orientation are determined using the approach proposed in [8].

Using vector function (70), we may determine projection \( L_L \) of the path of contact on the gear tooth surface on the tangent plane at the mean contact point \( M \). The goal is to obtain \( L_L \) as a straight line directed longitudinally. The process of computation is an iterative process and requires simultaneous application of local synthesis and TCA. The errors of alignment in this procedure are not taken into account. The influence of errors of alignment on transmission errors requires additional application of TCA.

It may be discovered in some cases of design, that the gear drive is too sensitive to errors of alignment. Then, it becomes necessary to deviate \( L_L \) from the longitudinal directed straight line by correction \( \eta_2 \) in procedure 3 (see Flow chart of Fig. 18).

8. Numerical Example

The theory developed will now be illustrated with the numerical example. The algorithms described above have been implemented by the computer program developed in Fortran and Visual Basic languages.

The blank data and gear machine-tool settings have been adapted from the Gleason summary and represented in Tables 2 and 3.

The final machine-tool settings for pinion generation have been determined as the output of the computer programs developed and are shown in Table 4. The results of TCA obtained by the developed computer programs are shown by Figs. 23, 24, 25, 26, and 27 for aligned and misaligned gear drives.

The investigation accomplished confirms that the longitudinal path of bearing contact developed is stabilized and the predesigned parabolic function absorbs the transmission errors caused by misalignment. The magnitude of transmission errors caused by misalignment might be controlled and does not exceed 7 arc seconds in the example.

### Table 2: Blank Data

<table>
<thead>
<tr>
<th>Applied parameters</th>
<th>Pinion</th>
<th>Gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth ( N_1, N_2 )</td>
<td>10</td>
<td>52</td>
</tr>
<tr>
<td>Shaft angle ( \gamma )</td>
<td>90°</td>
<td></td>
</tr>
<tr>
<td>Mean spiral angle ( \beta_1, \beta_2 )</td>
<td>32°30'</td>
<td>32°30'</td>
</tr>
<tr>
<td>Hand of spiral</td>
<td>Right hand</td>
<td>Left hand</td>
</tr>
<tr>
<td>Whole depth (mm)</td>
<td>16.637</td>
<td>16.637</td>
</tr>
<tr>
<td>Pitch angles ( \gamma_1, \gamma_2 )</td>
<td>10°53'</td>
<td>79°7'</td>
</tr>
<tr>
<td>Root angles ( \gamma_{r1}, \gamma_{r1} )</td>
<td>9°42'</td>
<td>75°8'</td>
</tr>
<tr>
<td>Face angles ( \gamma_{f1}, \gamma_{f1} )</td>
<td>14°52'</td>
<td>80°18'</td>
</tr>
<tr>
<td>Addendum (mm)</td>
<td>11.811</td>
<td>3.429</td>
</tr>
<tr>
<td>Dedendum (mm)</td>
<td>4.826</td>
<td>13.208</td>
</tr>
<tr>
<td>Face width ( F_w ) (mm)</td>
<td>71.120</td>
<td></td>
</tr>
<tr>
<td>Diametral pitch (1/inch)</td>
<td>3.156</td>
<td></td>
</tr>
<tr>
<td>Clearance ( c_1, c_2 ) (mm)</td>
<td>1.400</td>
<td></td>
</tr>
<tr>
<td>Mean cone distance ( A_m ) (mm)</td>
<td>177.521</td>
<td></td>
</tr>
</tbody>
</table>
Table 3: Gear Machine Tool Settings and Installment of Head-Cutter

<table>
<thead>
<tr>
<th>Applied parameters</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Blade Specifications</td>
<td></td>
</tr>
<tr>
<td>Blade angle at mean point $M$</td>
<td>$20^\circ$</td>
</tr>
<tr>
<td>Parabolic coefficient $a_c$</td>
<td>0.002</td>
</tr>
<tr>
<td>Cutter Radial $R_o$ (mm)</td>
<td>177.800</td>
</tr>
<tr>
<td>Cutter point radius $R_p$ (mm)</td>
<td>$177.800 \pm 2.4123$</td>
</tr>
<tr>
<td>Blade edge radius $\rho$ (mm)</td>
<td>2.794</td>
</tr>
<tr>
<td>Point width $P_w$ (mm)</td>
<td>4.826</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Gear Machine-Tool Settings</td>
<td></td>
</tr>
<tr>
<td>Machine root angle $\gamma_m$</td>
<td>$76^\circ 10'$</td>
</tr>
<tr>
<td>Machine center to back $X_G$ (mm)</td>
<td>-2.007</td>
</tr>
<tr>
<td>Horizontal $H_2$ (mm)</td>
<td>100.152</td>
</tr>
<tr>
<td>Vertical $V_2$ (mm)</td>
<td>137.897</td>
</tr>
</tbody>
</table>

Table 4: Pinion Machine-Tool Settings and Blade Data

<table>
<thead>
<tr>
<th>Applied parameters</th>
<th>Concave Side</th>
<th>Convex Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade angle $\alpha_p$</td>
<td>$17^\circ$</td>
<td>$23^\circ$</td>
</tr>
<tr>
<td>Cutter point radius $R_p$ (mm)</td>
<td>174.496</td>
<td>185.151</td>
</tr>
<tr>
<td>Radial setting $S_i$ (mm)</td>
<td>150.301</td>
<td>195.018</td>
</tr>
<tr>
<td>Installment angle $q_i$</td>
<td>$62,345''$</td>
<td>$61,3212''$</td>
</tr>
<tr>
<td>Machine center to back $X_{D_1}$ (mm)</td>
<td>-4.530</td>
<td>8.134</td>
</tr>
<tr>
<td>Sliding base $X_{B_1}$ (mm)</td>
<td>0.339</td>
<td>-1.795</td>
</tr>
<tr>
<td>Blank offset $E_m$ (mm)</td>
<td>19.947</td>
<td>-20.314</td>
</tr>
<tr>
<td>Ratio of cutting $m_{1p}$</td>
<td>4.853</td>
<td>5.773</td>
</tr>
<tr>
<td>Machine root angle $\gamma_m$</td>
<td>$9.7^\circ$</td>
<td>$9.7^\circ$</td>
</tr>
<tr>
<td>Coefficient of modified roll $b_2$</td>
<td>0.0302</td>
<td>-0.0572</td>
</tr>
<tr>
<td>Coefficient of modified roll $b_3$</td>
<td>-0.0599</td>
<td>0.1157</td>
</tr>
</tbody>
</table>
9. Conclusion

Based on the research performed the following conclusions can be drawn:

(1) The developed approach of design has been successfully applied and enabled to design formate cut spiral bevel gears with a stabilized bearing contact and favorable shape of transmission errors of low magnitude.

(2) The theoretical results obtained have been confirmed by simulation of meshing.

References

Appendix 1:—Relationships Between Principal Curvatures and Directions of Mating Surfaces

Introduction

The procedure of local synthesis requires the knowledge of principal curvatures and directions of contacting surfaces. In the case of design discussed in this paper, the pinion tooth surface is represented by three related parameters and the determination of pinion principal curvatures and directions is a complex problem. The solution to the problem is simplified by representation of pinion principal curvatures and directions in terms of principal curvatures and directions of the head-cutter and parameters of motions of the generating process (proposed in [8, 13]).

Important relations between principal curvatures and directions of mating surfaces in point contact (proposed in [8, 13]) are applied for local synthesis (see section 6).

Henceforth two types of instantaneous contact of meshing surfaces will be considered: (i) along a line, and (ii) at a point. Line contact is provided in meshing of the surface being generated with the tool surface. Point contact is provided for the generated pinion and gear tooth surfaces. The determination of the required relationships is based on the approach proposed in [8, 14]. The basic equations in the approach developed are as follows

\[ v_r^{(2)} = v_r^{(1)} + v^{(12)} \]  
\[ n_r^{(2)} = n_r^{(1)} + \omega^{(12)} \times n \]  
\[ \frac{d}{dt} [n \cdot v^{(12)}] = 0 \]

Equations (72) and (73) relate the velocities of the contact point and the tip of the unit normal in their motions over the contacting surfaces. Equation (74) represents the differentiated equation of meshing. Equations (72)-(74) yield a skew-symmetric system of three linear equations in two unknowns \( x_1 \) and \( x_2 \) of the following structure

\[ a_{ii} x_1 + a_{ij} x_2 = a_{ij} \quad (i=1-3) \]

Here: \( x_1 \) and \( x_2 \) are the projections of the velocity of the contact point in the motion over one of the surface on the principal directions of the mating surface. In case of line contact of surfaces the solution for the unknowns is indefinite and the rank of the system matrix of the linear equations is one. In case of point contact of surfaces the solution for the unknowns is definite, and the rank of the system matrix is two. The properties mentioned above are used for the derivation of the sought for relationships between the principal curvatures and directions of the meshing surfaces.

Meshing of Surfaces \( \Sigma_1 \) and \( \Sigma_2 \) : Surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are in point contact and their meshing is considered in fixed coordinate system \( S_t \) (Fig. 20). Equations (72)-(74) yield the following system of three linear equations [8, 14]

\[ a_{ii} v_1^{(1)} + a_{ij} v_q^{(1)} = a_{ii} \quad (i=1-3) \]

where
\[ v_s^{(1)} = v_r \cdot e_r, \quad v_q^{(1)} = v_r \cdot e_q \]  

\[
\begin{align*}
    a_{11} &= k_s - k_f \cos^2 \sigma^{(12)} - k_h \sin^2 \sigma^{(12)} \\
    a_{12} &= a_{21} = 0.5(k_f - k_h) \sin 2\sigma^{(12)} \\
    a_{13} &= a_{31} = -k_s v_s^{(12)} + [(n \times o^{(12)}) \cdot e_j] \\
    a_{22} &= k_q - k_f \sin^2 \sigma^{(12)} - k_h \cos^2 \sigma^{(12)} \\
    a_{23} &= a_{32} = -k_q v_q^{(12)} + [(n \times o^{(12)}) \cdot e_j] \\
    a_{33} &= k_s(v_s^{(12)})^2 + k_q(v_q^{(12)})^2 - [(n \times o^{(12)}) \cdot v^{(12)}] \\
                   &- n \cdot [(o^{(1)} \times v_r) - (o^{(2)} \times v_r^{(1)})] + m_ {21}^r(n \times k_j) \cdot r
\end{align*}
\]

The following are considered as being known: point \( M \) of tangency of surfaces \( \Sigma_1 \) and \( \Sigma_2 \), the common surface unit normal, the relative velocity \( v^{(12)} \), the principal curvatures \( k_s \) and \( k_q \) and directions \( e_s \) and \( e_q \) on \( \Sigma_2 \) at \( M \), and the elastic deformation \( \delta \) of surfaces at \( M \). The goal is to determine the principal curvatures \( k_f \) and \( k_h \) and the angle \( \sigma^{(12)} \) formed by the unit vectors \( e_r \) and \( e_s \).

The velocity \( v_r^{(i)} (i = l, 2) \) of the contact point on surface \( \Sigma_i \) has a definite direction and therefore the rank of the system matrix (76) is one. This property yields the following relation:

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
= F(k_f, k_h, k_s, k_q, \sigma^{(12)}, m_ {21}^r) = 0
\]

The sought for solution for \( k_f, k_h \) and \( \sigma^{(12)} \) can be obtained if the following parameters will be chosen: the derivative \( m_ {21}^r \); the ratio \( a/\delta \), where \( a \) is the major axis of the contact ellipse; the direction at \( M \) of the tangent to the contact path on one of the contacting surfaces \( \Sigma_1 \) and \( \Sigma_2 \). The relation between the directions at \( M \) of the tangents to the contact paths on both surfaces is represented by the equation [8, 14]

\[
\tan \eta_i = \frac{-a_ {33} v_q^{(12)} + (a_{33} + a_{33} v_s^{(12)}) \tan \eta_2}{a_{33} + a_{33}(v_q^{(12)} - v_s^{(12)}) \tan \eta_2}
\]

Choosing \( \eta_2 \) at point \( M \), we can determine \( \eta_1 \).

**Procedure of Determination of \( k_f, k_h \) and \( \sigma^{(12)} \)**

**Step 1:** Determine \( \eta_1 \) choosing \( \eta_2 \).

**Step 2:**

\[
v_r^{(1)} = \frac{a_{33}}{a_{13} + a_{23} \tan \eta_1}
\]
\[ v_q^{(i)} = \frac{a_{11} \tan \eta_i}{a_{11} + a_{21} \tan \eta_i} \]  
\( \text{(82)} \)

Step 3:
\[ A = \frac{\delta}{a} \]  
\( \text{(83)} \)

Step 4:
\[ K_{\Sigma} = \frac{a_{13} v_x^{(i)} + a_{23} v_y^{(i)}}{(v_x^{(i)})^2 + (v_y^{(i)})^2} + 2A \]  
\( \text{(84)} \)

Step 5:
\[ \begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} \frac{a_{13} v_x^{(i)} - a_{23} v_y^{(i)}}{(v_x^{(i)})^2 + (v_y^{(i)})^2} + (v_y^{(i)})^2 K_{\Sigma} \\ \frac{a_{13} v_x^{(i)} + a_{23} v_y^{(i)} - v_x^{(i)} v_y^{(i)} K_{\Sigma}}{(v_x^{(i)})^2 + (v_y^{(i)})^2} \\ -a_{13} v_x^{(i)} + a_{23} v_y^{(i)} + (v_y^{(i)})^2 K_{\Sigma} \end{bmatrix} \]  
\( \text{(85)} \)

Step 6:
\[ \tan 2\sigma^{(12)} = \frac{2a_{12}}{g_2 - (a_{11} - a_{22})} \]  
\( \text{(86)} \)

where \( g_2 = k_s - k_q \).

Step 7:
\[ g_1 = \frac{2a_{12}}{\sin 2\sigma^{(12)}} \]  
\( \text{(87)} \)

Step 8:
\[ K_{\Sigma}^{(1)} = K_{\Sigma}^{(2)} - K_{\Sigma} \]  
\( \text{(88)} \)

where \( K_{\Sigma}^{(2)} = k_s + k_q \).

Step 9:
\[ k_f = (K_{\Sigma}^{(1)} + g_1)/2 \]  
\( \text{(89)} \)

Step 10:
\[ k_h = (K_{\Sigma}^{(1)} - g_1)/2 \]  
\( \text{(90)} \)

The procedure provided above can be used to obtain the sought for principal curvatures \( k_f \) and \( k_h \) at point \( M \) of tangency of surfaces \( \Sigma_i \) and \( \Sigma_\rho \) and the principal directions on \( \Sigma_i \) at \( M \).

**Meshing of Surfaces \( \Sigma_i \) and \( \Sigma_\rho \):** The tool surface \( \Sigma_\rho \) generates the pinion tooth surface \( \Sigma_i \). Surfaces \( \Sigma_i \) and \( \Sigma_\rho \) are in line contact and point \( B \) is the given point of the instantaneous line of contact. The meshing of surfaces is considered in \( S_m \). At point \( B \) the following are assumed as given: the curvatures \( k_f \) and \( k_h \) of surface \( \Sigma_\rho \); the unit vectors \( e_i \) and \( e_h \) of principal directions on \( \Sigma_i \); the surfaces unit normal \( n \); the relative velocity \( v^{(12)} \). The goal is to determine the principal curvatures \( k_p \) and \( k_q \) of surface \( \Sigma_\rho \), and the angle \( \sigma^{(1\rho)} \) that is formed by the unit vectors \( e_i \) and \( e_\rho \).

Equation (72)-(74) yield a system of three linear equations.
\[ d_i v_i^{(p)} + d_2 v_h^{(p)} = d_{13} \quad (i=1-3) \]  

(91)

where

\[ v_i^{(p)} = v_{ri}^{(p)} \cdot e_r, \quad v_h^{(p)} = v_{rh}^{(p)} \cdot e_h \]  

(92)

The direction of \( v_i^{(p)} \) is indefinite since surfaces \( \Sigma_r \) and \( \Sigma_h \) are in line contact. Therefore, the rank of system matrix of equations is equal to one. Using this property, the following equations are obtained:

\[
\begin{aligned}
\tan 2\sigma^{(1p)} &= \frac{-2d_1 d_{23}}{d_{23}^2 - d_{13}^2 - (k_f - k_h)d_{33}} \\
k_r - k_p &= \frac{-2d_1 d_{23}}{d_{33} \sin 2\sigma^{(1p)}} \\
k_r + k_p &= k_f + k_h + \frac{d_{13}^2 + d_{23}^2}{d_{33}} 
\end{aligned}
\]  

(93)

Here:

\[
\begin{aligned}
d_{13} &= -k_f v_i^{(1p)} + [(n \times \omega_i^{(1p)}) \cdot e_r] \\
d_{23} &= -k_h v_h^{(12)} + [(n \times \omega_h^{(1p)}) \cdot e_h] \\
d_{33} &= -k_f (v_f^{(1p)})^2 - k_h (v_h^{(1p)})^2 - [(n \times \omega_f^{(1p)}) \cdot v_f^{(1p)}] \\
&\quad - n \cdot [(\omega_r^{(1p)} \times v_r^{(p)}) - (\omega_f^{(1p)} \times v_r^{(1)})] 
\end{aligned}
\]  

(94)

where

\[ v_f^{(1p)} = v^{(1p)} \cdot e_f, \quad v_h^{(1p)} = v^{(1p)} \cdot e_h \]  

(95)
Figure 1.—Illustration of tangency of surfaces $\Sigma_1$ and $\Sigma_2$. 
Figure 2.—Transmission function (a) and function of transmission errors (b) of a misaligned gear drive.
Figure 3.—Transmission function (a) and function of transmission errors (b) of a gear drive with a predesigned parabolic function.
Figure 4.—Interaction of pre-designed parabolic function $\Delta \phi_2^{(1)}(\phi_1)$ with linear function $\Delta \phi_2^{(2)}(\phi_1)$ caused by misalignment.
Figure 5.—Machine tool settings for gear generation. (a) For left hand gear. (b) For right hand gear.
Figure 6.—Illustration of gear cutter blades and generating surfaces.
Figure 7.—Geometric model of gear tooth surfaces.

Figure 8.—Machine tool settings for installment of the to-be-generated pinion.
Figure 9.—Machine tool settings for installment of pinion head-cutter. (a) For right hand pinion. (b) For left hand pinion.
Figure 10.—Illustration of pinion cutter blades and generating surfaces.
Figure 11.—Geometric model of pinion tooth surfaces.

Figure 12.—Illustration of parameters $\eta_2$ and $a$ applied for local synthesis.
Figure 13.—Projections of various paths of contact $L_T$ on tangent plane $T$. 

Tangent plane $T$ at point $M$
Figure 14.—Flow chart for procedure 1: design of bearing contact.
Figure 15.—Illustration of computations for determination of $\beta_2 (m_{12}') = 0$. 

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Figure 16.—Flow chart for procedure 2.

Start

Gear machine tool settings

Pinion machine tool settings

Modified roll

Gear tooth surface

Pinion tooth surface

TCA: Simulation of meshing and contact

Check if: Parabolic function of transmission errors desirable?

Yes!

Output predesigned parabolic function of transmission errors

No!

Correction of coefficients of modified roll

End
Figure 17.—Illustration of variation of coefficients $b_2$ and $b_3$ of modified roll.
Figure 18.—Flow chart for procedure 3.
Figure 19.—Representation of point $A$ in coordinate system $S_2$. 
Figure 20.—Coordinate systems $S_2$, $S_i$, and $S_1$ applied for local synthesis.

Figure 21.—For derivation of pinion machine tool settings.
Figure 22.—Coordinate system applied for simulation of meshing of pinion and gear tooth surfaces.
Figure 23.—TCA results output by Visual Basic: simulation of meshing without misalignment.
Figure 24.—TCA results output by Visual Basic: simulation of meshing with misalignment $\Delta H = 0.1$ mm.
Figure 25.—TCA results output by Visual Basic: simulation of meshing with misalignment $\Delta D = 0.1$ mm.
Figure 26.—TCA results output by Visual Basic: simulation of meshing with misalignment $\Delta E = 0.1$ mm.
Figure 27.—TCA results output by Visual Basic: simulation of meshing with misalignment $\Delta \gamma = 3'$.
Final Contractor Report

Computerized Design, Generation, and Simulation of Meshing and Contact of Face-Milled Formate Cut Spiral Bevel Gears

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A new approach for design, generation, and computerized simulation of meshing and contact of face-milled, formate cut spiral bevel gears is presented. The purpose is to develop a low noise, stabilized bearing contact for this type of gear drives. The approach proposed is based on application of three procedures that permit in sequence, to provide a longitudinally directed bearing contact, a predesigned parabolic function of transmission errors and limit the shift of bearing contact caused by errors of alignment. The theory developed is illustrated with an example of design and computation.