CHAPTER TWO: ALGEBRAIC OPERATIONS

COURSE TERMINAL OBJECTIVE:

Given a calculator, apply the basic concepts and principles of arithmetic, algebra, trigonometry, and calculus to perform mathematical analyses and solve problems. Proficiency is demonstrated by achieving 70% or better on written examinations which sample lesson objectives.

LESSON TERMINAL OBJECTIVE:

Given a calculator, apply the basic concepts and principles of algebra, as outlined in the lesson. Proficiency is demonstrated by achieving 70% or better on a written examination which samples lesson objectives.

LESSON ENABLING OBJECTIVES:

E01: Given the values of the literal numbers in an algebraic expression, evaluate the expression.

E02: Add, subtract, multiply and divide polynomial expressions.

E03: Solve linear equations in one unknown.

E04: Solve literal equations (formulas) for the literal number specified.

E05: Given two linear equations in two unknowns, determine the values of the unknowns.

E06: Apply the laws of exponents to simplify algebraic expressions containing exponents.

E07: Evaluate negative powers of real numbers.

E08: Evaluate fractional powers of numbers.

E09: Solve equations involving a single radical of a literal expression.

E10: Solve exponential equations which can be expressed as powers of like bases.

E11: Factor an algebraic expression by removing its greatest common factor.

E12: Factor the difference of two squares.

E13: Solve quadratic equations by use of the quadratic formula.

E14: Express algebraic fractions involving physical units in simplest form.

2.1 ALGEBRA

The algebra portion of the Mathematics Module is a part of the essential mathematics background required in order to perform the calculations and mathematical reasoning expected of nuclear reactor operators. In effect, you have been exposed to some of the elementary concepts and uses of algebra during the Arithmetic Refresher such as Set of Integers, Absolute and Algebraic Values, Signed Numbers and Symbols of Inclusion. In this course you will become familiar with elementary algebraic definitions, and laws and methods for solving intermediate level equations using algebraic principles.

DEFINITIONS

Algebra - A generalization of arithmetic in which symbols represent members of a specified set of numbers and are related by operations that hold for all numbers in the set.

Algebraic Sum - The sum of algebraic quantities produced by arithmetic addition, in which negative quantities may be added by the subtraction of corresponding positive quantities.

Algebraic Term - A product containing a literal number, i.e., $m = 1 \bullet m$, $xy = x \bullet y$, $2m = 2 \bullet m$, $6a^2 = 6 \bullet a^2$. Numerals are also considered to be terms; for example, 14 is a term.

Algebraic Expression - Sum, difference, product or quotient of one or more algebraic terms; may contain terms which are arithmetic numbers.

Monomial - An algebraic expression consisting of only one term.

Polynomial - An algebraic expression consisting of two or more terms, each separated from the next by a plus or minus sign.

EXAMPLE 1

If a nuclear reactor had 18 months to operate before refueling and spent 2X months at full load and Y months for testing, the time remaining to operate would be 18 - 2X - Y months.

a) 2X, like Y is an algebraic expression of one term, MONOMIAL

b) 2X + Y is an algebraic expression of two terms, BINOMIAL

c) 18 - 2X - Y is an algebraic expression of three terms, TRINOMIAL

d) $2X + Y$ and $18 - 2X - Y$ each contain more than one term; each is a POLYNOMIAL.

Numerical Coefficient - The arithmetic factor of an elementary algebraic term, i.e., in the term $2X$, the number 2 is the numerical coefficient; and in the term $x/2$, $1/2$ is the numerical coefficient.

Constant - A quantity which keeps the same value.

Variable - A quantity whose value may change.

Equation - A statement of equality between two expressions. An equation always contains a left member, an equals sign and a right member.

Function - A function is a correspondence between two sets of numbers which assigns to each member of the first set one and only one member of the second set. The first set is called the DOMAIN of the function and the second set is called the RANGE of the function. If x is a variable representing elements of the domain, then x is called the INDEPENDENT VARIABLE. If y is the variable representing elements of the range of the function, then y is called the DEPENDENT VARIABLE. Therefore, a function is a rule which assigns to each value of the independent variable a unique value of the dependent variable.

In the equation $y = x^2$, y is a function of x because for each value of x one obtains a unique value of y. However, x is not a function of y, because for each value of y one obtains two values of x. For example, if $y = 4$, then x may be 2 or -2.

EXAMPLE 2

To convert horsepower (mechanical power) to units of electrical power, the following equation may be used. (hp represents horsepower and W represents watts)

hp = 746 W (equation)
hp = dependent variable and is a function of W
W = independent variable
746 = numerical coefficient of W and is a constant

Formula - A literal equation that states a general rule such as the equation in example 2.

Similar algebraic terms - Like terms; the literal (x,y,a,b) parts of the terms are exactly alike.

EXAMPLE 3

The top of a desk is x inches wide and 2x inches long. The number of square inches (area) in the desk top is $x \bullet 2x$ or $2x^2$.

a) x and 2x are similar or like terms.

b) 2x and $2x^2$ are NOT similar terms because their literal parts are not the same.

1.2.1 FUNDAMENTAL OPERATIONS.

The operations performed with algebraic terms require a strict obedience to the fundamental laws of algebra.

2.1.1 COMBINING TERMS AND EVALUATING EXPRESSIONS.

a) Similar or like terms can be combined by addition and subtraction; unlike terms cannot.

b) To ADD or SUBTRACT similar terms, add or subtract their numerical coefficients and multiply the result by the common literal factor.

EXAMPLE 4

5x and 3x may be combined by addition: $5x + 3x = 8x$, or subtraction: $5x - 3x = 2x$; 2x and $2x^2$ cannot be combined by addition or subtraction.

EXAMPLE 5

Suppose that a prospective reactor operator spends h hours on-the-job training, $\frac{h}{2}$ hours formal classroom instruction and $\frac{h}{3}$ hours home study.

TOTAL TIME ACCOUNTED FOR:

$$h + \frac{h}{2} + \frac{h}{3} = 1 \bullet h + \frac{1}{2} h + \frac{1}{3} h$$

$$= \frac{6}{6} h + \frac{3}{6} h + \frac{2}{6} h$$ (common denominator so that the numerical coefficients can be combined)

$$= \frac{11}{6} h$$ (combine terms by addition)

THE DIFFERENCE BETWEEN THE TIME SPENT IN CLASS AND THE TIME FOR HOME STUDY:

$$\frac{h}{2} - \frac{h}{3} = \frac{1}{2}h - \frac{1}{3}h$$

$$= \frac{3h}{6} - \frac{2h}{6} \quad \text{(common denominator)}$$

$$= \frac{1}{6}h \quad \text{(combine by subtraction)}$$

c) Symbols of inclusion are used in polynomials to show that terms are all part of one expression. AN EXPRESSION WITHIN A SYMBOL OF INCLUSION IS CONSIDERED A SINGLE TERM.

<u>EXAMPLE 6</u>

In the expression $4 + (x + 1)$; $(x + 1)$ is considered a single term.

d) The general order of operations is powers first, multiplications and divisions second, then additions and subtractions.

e) To evaluate an algebraic expression, arithmetic numbers may be substituted for each letter or unknown; then find the value of the expression by following the rules for order of operations.

<u>EXERCISES</u>

1) Combine <u>similar</u> terms in each expression. DO <u>NOT</u> perform multiplications or divisions.

a) $6x - 7 + 8x$

b) $7y + 8 - 3y$

c) $11a - 4b - 10a + 8b$

d) $5m + 4n - 3m + 2n$

e) $6x^2 + 4x - 3x^2 - 15$

f) $5b^3 - 6 + b^3 + 8$

g) $4a^3 + 2b^2 - 2a^3 + b - 4$

h) $m - n + .9m - .9n$

i) $30(a + b) - 8(a + b) + 5ab$

j) $6x^2y - xy^2 - 6 + 3x^2y + xy^2 + 8$

k) $3x^2 + 5x$

2) Evaluate the following expressions if $a = 1$, $b = 2$, $c = 3$ and $d = 4$.

a) $2a + b$

b) $7a - 2b$

c) ab^2c

d) abc^2

e) $d^2 - a^2$

f) $c^2 - d^2$

g) $bd - (a + c)$

h) $ac - (c - a)$

i) $\frac{d}{2b} + (bc)^2$

j) $(bd)^2 - \frac{3d}{4c}$

k) $\frac{3d - b}{5} + 2c^2$

l) $3b^2 - \frac{4c - d}{4}$

2.1.2 ADDITION AND SUBTRACTION OF POLYNOMIALS

The addition of polynomials may be performed vertically, with similar terms written beneath each other. The columns are then added one by one, the sign of the term being considered as part of the column.

EXAMPLE 7

Find the sum of $(x^2 + 2x + 1) + (2x^2 - 8) + (6 - 5x + x^2)$

$$\begin{array}{r|r|r l} x^2 & +2x & +1 & \text{(addend)} \\ 2x^2 & & -8 & \text{(addend)} \\ x^2 & -5x & +6 & \text{(addend)} \\ \hline 4x^2 & -3x & -1 & \text{(sum)} \end{array}$$

Another method is to remove the parentheses and combine similar terms. IF THE GROUP WITHIN A SYMBOL OF INCLUSION IS PRECEDED BY A PLUS SIGN, THE SYMBOL OF INCLUSION MAY SIMPLY BE REMOVED.

EXAMPLE 8

$(x^2 + 2x + 1) + (2x^2 - 8) + (6 - 5x + x^2)$

$x^2 + 2x + 1 + 2x^2 - 8 + 6 - 5x + x^2$ (Remove parentheses)

$\underbrace{x^2 + 2x^2 + x^2}_{4x^2} + \underbrace{2x - 5x}_{-3x} + \underbrace{6 + 1 - 8}_{-1}$ (Group similar terms) (Combine similar terms)

The subtraction of polynomials may be performed vertically; the terms of the subtrahend are written beneath similar terms of the minuend, and the columns are subtracted one by one. REMEMBER THE LAWS OF SIGNED NUMBERS, i.e., change the sign of the subtrahend and add algebraically.

EXAMPLE 9

Find the difference $(a^2 - 2ab + b^2) - (b^2 - ab + 3a^2)$

$$\begin{array}{r|r|r l} a^2 & -\ 2ab & +\ b^2 & \text{(minuend)} \\ -\ 3a^2 & +\ ab & -\ b^2 & \text{(subtrahend with signs changed)} \\ \hline -\ 2a^2 & -\ ab & 0 & \text{(difference)} \end{array}$$

or

$$-2a^2 - ab$$

Another method is to remove the parentheses and combine similar terms. IF THE GROUP WITHIN A SYMBOL OF INCLUSION IS PRECEDED BY A MINUS SIGN, CHANGE THE SIGN OF ALL TERMS INSIDE AND REMOVE THE GROUPING SYMBOL AND PRECEDING MINUS SIGN. This rule may also be stated as, TO REMOVE THE GROUPING SYMBOL AND PRECEDING MINUS SIGN, MULTIPLY ALL TERMS INSIDE BY -1.

$$-(x + y) = -1\ (x + y) = (-1)\ (x) + (-1)(y) = -x - y$$

EXAMPLE 10

$(a^2 - 2ab + b^2) - (b^2 - ab + 3a^2)$

$-1(b^2 - ab + 3a^2)$ (Multiply all terms inside

$-1(+b^2) - 1(-ab) - 1\ (+3a^2)$ by -1 and remove parentheses)

$a^2 - 2ab + b^2 \quad -b^2 \quad + ab \quad - 3a^2$

$a^2 - 3a^2 - 2ab + ab + b^2 - b^2$ (Group similar terms)

$-2a^2 \quad -ab \quad 0$ (Combine similar terms)

$-2a^2 - ab$

EXERCISES

Perform the indicated additions and subtractions:

1) $(5x^3 - 7x^2 + 3x + 1) + (3x^3 - 5x + 18 + x^2)$

2) $(7a^4 - 3a^3 + 5a^2 + 2) + (3a^2 - 4a^3 + 8a^4 - 3)$

3) $(2x^3y - 3x^2y^2 - 5\ xy^3) + (x^2y^2 - x^3y)$

4) $(-5a^5b + 8b^4 - a^3b^2) + (5b^4 + 5a^5b)$

5) $(6n^3 - 5n - 3) + (8n^3 + n^2 - 7) + (3n^3 + 2n^2 + n - 1)$

6) $[6(a + b)^2 - 2(a + b)^3 + 11] + [7(a + b)^2 + 2(a + b)^3 - 10]$

7) $(7x^3 - 8x^2 - 2x - 6) - (5x^3 - 2x + 5 - x^2)$

8) $(8a^4 - 3a^3 + 7a^2 + a - 6) - (9a^3 - a^4 + a^2 + 3)$

9) $(x^3 - 3x^2 + 4x^4 - 6) - (3x^4 - 2x^3 - 8x - 10)$

10) $(a^4 + a^3 - 7a + 5) - (3a + 2a^2 + a^3 - 6)$

11) $[6(x + y)^2 - 5(x + y) - 8] - [5(x + y)^2 - 4(x + y) - 9]$

12) $[3(n - b)^2 + 2(n - b) + 7] - [2(n - b)^2 - (n - b) + 8]$

2.1.3 MULTIPLICATION

The product of powers having the SAME BASE may be found by adding the exponents, keeping the same base.

EXAMPLE 11

$x \cdot x^2 = x^{1+2} = x^3$ since $x \cdot x^2 = (x)(x \cdot x)$

$y^3 \cdot y^4 = y^{3+4} = y^7$ since $y^3 \cdot y^4 = (y \cdot y \cdot y)(y \cdot y \cdot y \cdot y)$

REMEMBER:
The exponent shows how many times the base is used as a FACTOR. A base without an exponent is understood to have the exponent 1; it is used as a factor once. A base with a zero for an exponent is equal to 1.

The PRODUCT of MONOMIALS may be found by FIRST multiplying the signed numerical coefficients, THEN multiplying the literal factors in ALPHABETICAL ORDER.

EXAMPLE 12

$(-5xy)(7x^2y) = (-5)(7)(x^{1+2})(y^{1+1}) = -35x^3y^2$

EXAMPLE 13

$(3x^2)(-2y)(-6y^3) = (3)(-2)(-6)(x^2)(y^{1+3}) = +36x^2y^4$

EXERCISES

Find the following products:

1) $a^3 \cdot a^2$

2) $x^3 \cdot x^3$

3) $x^a \cdot x^b$

4) $y^{x+1} \cdot y^{x-1}$

5) $3x^3 \cdot 8x^3$

6) $(-7xy)(-8x^2y)$

7) $(-5a^2b)(-3ab^2)$

8) $(-9abc)(3a^2b^3c)$

9) $(12rst^2)(-3r^2s^2t)$

10) $(-2a)(-3a^2)(-4a^3)$

11) $(-3xy)(-4x)(5y^3)$

12) $(2x^3)(x^{-3})(y^2)$

To multiply a POLYNOMIAL by a MONOMIAL, multiply EACH TERM OF THE POLYNOMIAL by the monomial and combine the products.

EXAMPLE 14

$$a(b + c) = ab + ac \quad \text{and,}$$

$$a(b - c) = ab - ac$$

The above example expresses what is known as the Distributive Law of multiplication with respect to addition and subtraction; the multiplier, a, is distributed to each term of the multiplicand.

EXAMPLE 15

$$3x(x + 4y) = 3x(x) + 3x(4y) = 3x^2 + 12xy$$

EXAMPLE 16

$$2a(a^2 - 5ab - 6b^2) = 2a(a^2) + 2a(-5ab) + 2a(-6b^2) \quad \text{(multiply each term by +2a)}$$

$$= 2a^3 - 10a^2b - 12ab^2$$

EXERCISES

1) $a(3a^3 - 2a^2 + 5a)$

2) $-a(3a^3 - 2a^2 + 5a)$

3) $-3n(3n^2 - n + 2)$

4) $4x^2(12x^3 - 3x^2 + 5x - 1)$

5) $4n\left(\frac{1}{2} n^2 + \frac{1}{4}n - \frac{1}{8}\right)$

6) $12a\left(\frac{1}{12} a^2 - \frac{5}{6} a - \frac{7}{4}\right)$

To MULTIPLY TWO POLYNOMIALS, multiply each term in one factor by each term in the other factor and combine the products.

EXAMPLE 17

$$(a + b)(c + d) = ac + ad + bc + bd \quad \text{and,}$$

$$(a + b)(c - d) = ac - ad + bc - bd \quad \text{and,}$$

$$(a - b)(c - d) = ac - ad - bc + bd$$

EXAMPLE 18

$(x + 7)(2x - 3) = x(2x) + x(-3) + 7(2x) + 7(-3)$ (multiply each term of one by each term of the other)

$= 2x^2 - 3x + 14x - 21$

$= 2x^2 + 11x - 21$ (combine like terms)

The multiplication may also be arranged vertically, similar to multiplying whole numbers.

EXAMPLE 19

$$\begin{array}{rrl}
 & 2x - 3 & \text{(multiplicand)} \\
 & \underline{x + 7} & \text{(multiplier)} \\
 & 14x - 21 & \text{(multiply by +7)} \\
\underline{2x^2 - 3x} & & \text{(multiply by +x)} \\
2x^2 + 11x - 21 & & \text{(combine similar terms)}
\end{array}$$

EXERCISES

1) $(2x + 7)(3x - 8)$

2) $(5c - 4)(4c + 5)$

3) $(8a - b)(6a - 2b)$

4) $(m - n)(m^2 - n^2)$

5) $(2x + 3)(3x + 2)$

6) $(x + y)(x - y)$

When a POLYNOMIAL IS RAISED TO A POWER, the entire polynomial is used as a factor the number of times indicated by the exponent.

EXAMPLE 20

$(2n - 3)^2 = (2n - 3)(2n - 3) = 4n^2 - 6n - 6n + 9 = 4n^2 - 12n + 9$

$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

When multiplying several polynomials, consider the factors TWO AT A TIME.

<u>EXAMPLE 21</u>

$$\begin{aligned}(2a + 1)(a - 3)^2 &= (2a + 1)\underbrace{(a - 3)(a - 3)} \\ &= (2a + 1)(a^2 - 6a + 9) \\ &= 2a^3 - 12a^2 + 18a + a^2 - 6a + 9 \\ &= 2a^3 - 11a^2 + 12a + 9\end{aligned}$$

(Multiply <u>each</u> term by 2a + 1)

(combine terms)

<u>EXAMPLE 22</u>

$$\begin{aligned}(a + b)^3 &= (a + b)\underbrace{(a + b)(a + b)} \\ &= (a + b)(a^2 + 2ab + b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

(Multiply each term by a + b)

(combine terms)

In an expanded expression, terms containing powers of the same base are arranged in the order of either descending or ascending powers of that base.

<u>EXAMPLE 23</u>

$(a + 5)(a - 4) = a^2 + a - 20$ (descend)

$b(2 - b)(3 - b)(= b(6 - 2b - 3b + b^2) = 6b - 5b^2 + b^3$ (ascend)

If the expression contains more than one variable, arrange the expanded expression in either descending or ascending powers of one of the variables.

$(3a + b)^2 = (3a + b)(3a + b) = 9a^2 + 6ab + b^2$

<u>EXERCISES</u>

1) $(x + y)^2$

2) $(x - y)^2$

3) $(x - y)^3$

4) $(x + y)(x^2 - xy + y^2)$

5) $(x - y)(x^2 + xy + y^2)$

6) $(3x + 1)^2$

7) $(x^2 - y)^2$

8) $(r - 1)^3$

9) $(x + 1)(2x^2 + 1 - 3x + x^3)$

10) $(x - 1)^2 (x^2 + 2x + 1)$

2.1.4 DIVISION

The QUOTIENT OF TWO MONOMIALS may be found by writing the division as a fraction, then reducing the fraction to lowest terms; FACTOR THE TERMS of THE FRACTION and divide BOTH numerator and denominator by their common factors.

EXAMPLE 24

$$\frac{4ab^2c}{-12abc^2} = \frac{\overset{1}{\cancel{(4)}}\overset{1}{\cancel{(a)}}\overset{1}{\cancel{(b)}}(b)\overset{1}{\cancel{(c)}}}{\underset{-3}{\cancel{(-12)}}\underset{1}{\cancel{(a)}}\underset{1}{\cancel{(b)}}\underset{1}{\cancel{(c)}}(c)} = \frac{1 \cdot b}{-3 \cdot c} = -\frac{b}{3c}$$

EXAMPLE 25

$$\frac{3(ab)^3}{ab} = \frac{3\overset{1}{\cancel{(a \cdot b)}}(a \cdot b)(a \cdot b)}{\underset{1}{\cancel{(a \cdot b)}}} = \frac{3(a \cdot b)(a \cdot b)}{1} = 3(ab)^2$$

EXERCISES

1) $\frac{10a^3b}{5a^2b}$

2) $20r^2t \div 4rt$

3) $-12x^5 \div 3x^2$

4) $\frac{+14a^2b^3}{-7a^2b}$

5) $\frac{-75x^2y^3}{-25xy}$

6) $\frac{-96c^3d}{-16c^2d}$

To DIVIDE A POLYNOMIAL BY A MONOMIAL, divide EACH TERM of the polynomial by the monomial and combine the quotients.

EXAMPLE 26

$\frac{b + c}{d} = \frac{b}{d} + \frac{c}{d}$; also $\frac{b - c}{d} = \frac{b}{d} - \frac{c}{d}$

The above example expresses the Distributive Law of division with respect to addition and subtraction. The divisor, d, is distributed to each term of the dividend.

EXAMPLE 27

$$\frac{15x^2 + 30x}{5x} = \frac{15x^2}{5x} + \frac{30x}{5x} = 3x + 6$$

EXAMPLE 28

$$\frac{y^2 - y - 1}{y} = \frac{y^2}{y} - \frac{y}{y} - \frac{1}{y} = y - 1 - \frac{1}{y}$$

EXERCISES

1) $\dfrac{25x^3 + 15x^2 - 30x}{5x}$

2) $\dfrac{21a^4 - 14a^3 + 7a^2}{7a}$

3) $\dfrac{16x^3y - 24x^2y^2 - 64xy^3}{-8xy}$

4) $\dfrac{36a^4 + 18a^3b - 24a^2b^2}{-6a^2}$

5) $\dfrac{r^2s - r^2s^2 - 5rs^2}{-rs}$

6) $\dfrac{12a^3 - 9a^2 + 3a}{3a^2}$

2.1.5 PRACTICE PROBLEMS

Simplify by combining similar terms.

1) $9x + 2y - 5x$

2) $6.7a + 4.3b - 4.1a - 2.9b$

3) $\frac{7}{8} x + \frac{5}{9} y - \frac{1}{2} x - \frac{2}{3} y$

4) $9a^2b + 7ab^2 - 5 - 3a^2b - 7ab^2 + 6$

5) $12(a + b) - 2(a + b) - 5(a + b)$

Evaluate the following expressions by substitution, if $x = 2$, $y = 3$, and $z = 4$.

6) $3x - y + z$

7) x^2y^3

8) $z^2 - x^2$

9) $xz - (z - x)$

10) $(xy)^2 - \frac{4z}{2x}$

Evaluate the following by substitution, if $a = -2$, $b = 3$, and $c = -4$.

11) $(ac)^2$

12) $2a + c$

13) $ab^2 + c$

14) $\frac{3a - b}{3} + b^2c$

15) $4b^2 + \frac{4a + c}{4} + \sqrt[3]{9b}$

Remove the parentheses and simplify.

16) $9x - (4x + 3)$

17) $2a + (5a - 9)$

18) $2x - 4y - (7x - 2y)$

19) $5(x + 2) - 3x$

20) $[7(x + 5) - 19] - [4(x - 6) + 10]$

21) $3\{[6(x - 2) + 4] - [2(2x - 5) + 6]\}$

Perform the indicated additions and subtractions of polynomials.

22) $(8r^3 + 13r^2 - 6r + 7) + (r^2 - 3r^3 - r)$

23) $[2(x + y)^3 - (x + y) + 8] + [(x + y)^3 + (x + y) - 9]$

24) $(ax + by + c) + (3ax - 2by - c)$

25) $(cn^2 + dn - e) - (cn^2 - dn - e)$

26) $(3x^3y - 4x^2y^2 - 5xy^3) - (x^2y^2 - x^3y)$

27) Subtract the sum of $(x + a)$ and $(x + b)$ from $(3x + 2a - b)$

28) Find the difference between the sum of $(5x + y)$ and $(3x + 5y)$ and the sum of $(x - y)$ and $(x + y)$.

Find the following products:

29) $h^2 \cdot h^3 \cdot h^4$

30) $4n^3 \cdot 5n^2$

31) $2x^4 \cdot 5x^{-2}$

32) $y^{x + 1} \cdot y^{x - 1}$

33) $3x^{-2} \cdot 4x^5$

34) $(4x^2y)(- 3xy^2)$

35) $(- 3xyz)(5x^2y^3z)$

36) $(- a)(-2a^2)(- 4x^3)$

Multiply the following:

37) $x(5x^3 - 2x^2 + 3x)$

38) $- a(4a^4 - 2a^3 - 5a^2 + a)$

39) $5r^2(3r^2 - 2r + 1)$

40) $4x\left(\frac{1}{8}x^2 - \frac{3}{4}x - \frac{1}{16}\right)$

Multiply the following polynomials:

41) $(3x - 2)(2x - 3)$

42) $(x + 3a)(x - 2a)$

43) $(x^2 + 2)(x^2 - 2)$

44) $(x + 4)^2$

45) $(x - 3)(x + 3)^2$

46) $(x + 1)(3x^2 + x - 2)$

47) $(a - 1)^3$

48) $(2x^2 + 3x - 2)(x^2 - 2x + 3)$

49) $(5x^3 - 3x^2 + 2x - 1)(x + 1)$

50) $(a + 2)^3 (a + 1)$

Find the quotients to the following:

51) $\dfrac{4x^2y^2z}{2xy}$

52) $\dfrac{9c^5}{-9c^5}$

53) $\dfrac{b^2c^2}{-2b^2c^2}$

54) $(mn)^4 \div mn$

55) $\dfrac{3(ab)^3}{ab}$

Find at the indicated quotients.

56) $\dfrac{3a^4 - 3a^3 + 3a^2}{3a^2}$

57) $\dfrac{-15x^3 + 30x^2 + 5}{5x}$

58) $\dfrac{15(x + y)^3}{3(x + y)}$

59) $\dfrac{33r^5 + 22r^4 + 11}{11r^3}$

60) $8bc \div (-ac)$

2.2 LINEAR EQUATIONS AND FORMULAS

An algebraic equation is a mathematical sentence stating that two algebraic expressions are equal. An equation always contains a left member, an equals sign (=) and a right member. An UNKNOWN in an equation is a literal number whose value is to be found.

A LINEAR EQUATION is an equation in which the highest power of an unknown value is the first power; it is also called a FIRST DEGREE EQUATION. A linear equation may contain more than one unknown, but will always be in the first degree.

A SOLUTION or ROOT of an equation is a replacement, or substitute, for the variable that makes the equation a true statement; it causes the left member and right member of the statement to become identical.

Each step in the process of solving equations amounts to the replacement of the original equation by a simpler equation that is equivalent to it. For linear equations, the process amounts to isolating the terms involving the unknown on one side of the equation, and constants on the other side. The following basic procedures are valid ways to accomplish this:

1. Add the same quantity to both sides of the equation.
2. Subtract the same quantity from both sides of the equation.
3. Multiply both sides of the equation by the same non-zero quantity.
4. Divide both sides of the equation by the same non-zero quantity.

The following examples illustrate the four principles listed above.

EXAMPLE 1: Solve $x - 13 = 11$.

Solution: $x - 13 \underline{+ 13} = 11 \underline{+ 13}$ (Add 13 to both sides)

$x = 24$, answer.

EXAMPLE 2: Solve $x + 16 = 49$.

Solution: $x + 16 \underline{- 16} = 49 \underline{- 16}$ (Subtract 16 from both sides)

$x = 33$, answer.

EXAMPLE 3: Solve $\frac{x}{3} = -2$.

Solution: $3\left(\frac{x}{3}\right) = 3\ (-2)$ (Multiply both sides by 3)

$x = -6$, answer.

EXAMPLE 4: Solve $\frac{-3x}{5} = 9$.

Solution: $-\frac{5}{3}\left(\frac{-3x}{5}\right) = \left(-\frac{5}{3}\right)(9)$ (Multiply both sides by $-\frac{5}{3}$)

$x = -15$, answer.

EXAMPLE 5: Solve $3x = 81$.

Solution: $\frac{3x}{3} = \frac{81}{3}$ (Divide both sides by 3)

$x = 27$, answer.

EXAMPLE 6: Solve $-0.2x = 18.6$.

Solution: $\frac{-0.2x}{-0.2} = \frac{18.6}{-0.2}$ (Divide both sides by -0.2)

$x = -93$, answer.

EXAMPLE 7: Solve $3x - 5 = 11$.

Solution: $3x - 5 + 5 = 11 + 5$ (Add 5 to both sides)

$3x = 16$ (Simplify)

$\frac{3x}{3} = \frac{16}{3}$ (Divide both sides by 3)

$x = \frac{16}{3}$, answer.

<u>EXAMPLE 8</u>: Solve $3x + 7 = 5x - 9$.

Solution:

$$3x + 7 \underline{- 5x} = 5x - 9 \underline{- 5x} \quad \text{(Subtract 5x from both sides)}$$

$$-2x + 7 = -9 \quad \text{(Simplify)}$$

$$-2x + 7 \underline{- 7} = -9 \underline{- 7} \quad \text{(Subtract 7 from both sides)}$$

$$-2x = -16 \quad \text{(Simplify)}$$

$$\frac{-2x}{-2} = \frac{-16}{-2} \quad \text{(Divide both sides by -2)}$$

$x = 8$, answer.

REMEMBER: To solve linear equations, the basic idea is to isolate the terms involving the unknown on one side, and the constant terms on the other. Sometimes, basic algebraic operations must be performed before this process of isolation can begin. The following examples illustrate.

<u>EXAMPLE 9</u>: Solve $2(x-5) = 6$.

Solution:

$$2x - 10 = 6 \quad \text{(Distributive property)}$$

$$2x = 16$$

$x = 8$, answer.

EXAMPLE 10: Solve $3x - 5(2-2x) = 11 + 9x$

$$3x - 10 + 10x = 11 + 9x \quad \text{(Distributive property)}$$

$$13x - 10 = 11 + 9x \quad \text{(combine like terms)}$$

$$4x - 10 = 11$$

$$4x = 21$$

$x = \frac{21}{4}$, answer.

EXAMPLE 11: Solve $(2x + 4)(x - 3) = 2x^2 - 6x + 3$

Solution: $2x^2 - 6x + 4x - 12 = 2x^2 - 6x + 3$ (Multiply polynomials)

$$2x^2 - 2x - 12 = 2x^2 - 6x + 3$$

$$-2x - 12 = -6x + 3$$

$$4x - 12 = 3$$

$$4x = 15$$

$x = 15/4$, answer.

Sometimes linear equations involve fractions. As a general rule, it is best to multiply both sides by the lowest common denominator of those fractions first, so that the equation becomes free of fractions. The following examples illustrate.

EXAMPLE 12: Solve $\frac{2x}{3} - \frac{1}{5} = 2$

Solution: $15\left(\frac{2x}{3} - \frac{1}{5}\right) = 15(2)$ (Multiply both sides by 15)

$$10x - 3 = 30$$

$$10x = 33$$

$x = 33/10$, answer.

EXAMPLE 13: Solve $\frac{x-4}{5} - \frac{1}{2} = 3$

Solution: $10\left(\frac{x-4}{5} - \frac{1}{2}\right) = 10(3)$ (Multiply both sides by 10)

$$2(x-4) - 5 = 30$$

$$2x - 8 - 5 = 30$$

$$2x - 13 = 30$$

$$2x = 43$$

$x = 43/2$, answer.

<u>EXAMPLE 14</u>: Solve $\frac{x}{4} - \frac{3-2x}{2} = \frac{1}{3}$

Solution:

$$12\left(\frac{x}{4} - \frac{3-2x}{2}\right) = 12\left(\frac{1}{3}\right)$$

$$3x - 6(3-2x) = 4$$

$$3x - 18 + 12x = 4$$

$$15x - 18 = 4$$

$$15x = 22$$

$x = 22/15$, answer.

EXERCISES:

Solve the following equations.

1. $3x - 7 = -11$

2. $2(x-3) + 3(x-2) = x + 4$

3. $3x - 4(2x-5) = x - 9$

4. $\frac{5x}{3} = 60$

5. $2x(x-4) = (2x+3)(x-2)$

6. $\frac{x}{3} - \frac{1}{2} = -4$

7. $x - 2 = \frac{x+2}{4} + 5$

8. $\frac{y+3}{6} + 5 = \frac{2y-6}{3}$

9. $\frac{3y}{2} - \frac{y+3}{3} = 8 - \frac{y+2}{4}$

10. $\frac{3}{2} - \frac{3y-1}{3} = y - \frac{5}{6}$

2.3 FORMULAS

A formula is a general equation for the solution of a specific problem. Often a formula is not in the form desired or is not in a form convenient for the solution of a problem. Since a formula is an equation, it may be treated as such and changed to the desired equivalent equation by the methods shown in the preceding sections. To evaluate a formula, substitute the known values and solve the resulting equation.

EXAMPLE 1:

Solve the formula $P = 2L + 2W$ for W.

Solution: $P = 2L + 2W$

$P-2L = 2W$ (Subtract 2L from both sides)

$\frac{P-2L}{2} = W$ (Divide both sides by 2)

EXAMPLE 2:

Solve the formula $K = \frac{1}{2}mv^2$ for m

Solution: $K = \frac{1}{2}mv^2$

$2K = mv^2$ (Multiply both sides by 2)

$\frac{2K}{v^2} = m$ (Divide by both sides by v^2)

EXAMPLE 3:

Determine C, if $C = \frac{5}{9}(F-32)$ and $F = 41$

Solution: $C = \frac{5}{9}(F-32)$

$C = \frac{5}{9}(41-32)$ (Substitute $F = 41$)

$C = \frac{5}{9}(9)$

$C = 5$, answer.

EXAMPLE 4:

Determine c, if $T = \frac{b-r}{cr}$ and T = 700, $b = 7.2 \times 10^{-3}$, and $r = 2 \times 10^{-4}$.

Solution: There are two possible approaches here:

1. Solve the equation for c, then substitute known values.

2. Substitute known values, then solve resulting equation for c.

Either method is acceptable; both methods are shown.

Method 1: $T = \frac{b-r}{cr}$

$$Tcr = b-r \qquad \text{(Multiply both sides by cr)}$$

$$c = \frac{b-r}{Tr} \qquad \text{(Divide both sides by Tr)}$$

$$c = \frac{7.2 \times 10^{-3} - 2 \times 10^{-4}}{700\ (2 \times 10^{-4})}$$

c = 0.05, answer.

Method 2: $T = \frac{b-r}{cr}$

$$700 = \frac{7.2 \times 10^{-3} - 2 \times 10^{-4}}{c\ (2 \times 10^{-4})}$$

$$0.14c = 7.2 \times 10^{-3} - 2 \times 10^{-4}$$

$$c = \frac{7.2 \times 10^{-3} - 2 \times 10^{-4}}{0.14}$$

c = 0.05, answer.

EXERCISES

Solve the following formulas for the letter indicated:

1. $C = 2\pi r$, for r

2. $C = \frac{5}{9}(F-32)$, for F

3. $V = \frac{1}{3}\pi r^2 h$, for h

4. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$, for T_2

2.4 LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the equation $3x + 2y = 11$. This equation is called a linear equation in two unknowns because

1. It contains two unknowns, x and y
2. Both unknowns are raised to the first power.

How many pairs of numbers x and y make the equation true?

In fact, there are an infinite number of pairs of numbers that satisfy the equation. Examples of solutions include

$x = 1$ and $y = 4$

$x = 3$ and $y = 1$

$x = 5$ and $y = -2$, etc.

Any one equation in two unknowns is therefore indeterminate, i.e., it has many possible solutions.

Now consider the following pair of two equations in two unknowns:

$$\begin{aligned} 3x + 2y &= 11 \\ 2x + y &= 6 \end{aligned}$$

How many pairs of numbers x and y satisfy both equations simultaneously? Generally, there is only one pair which satisfy both equations. The discussion below illustrates two different techniques which can be used to determine the solution common to both equations.

SUBSTITUTION METHOD:

Steps:
1. Solve one of the equations for one of the unknowns in terms of the other unknown.
2. Substitute the result into the second equation and solve it.
3. Substitute the answer obtained into the first equation to determine the value of the second variable.

EXAMPLE 1:

Find the common solution to the following pair of equations in two unknowns using the substitution method:

$$3x + 2y = 11$$

$$2x + y = 6$$

Solution: Since the second equation is easy to solve for y, do this as step 1:

$$y = 6-2x.$$

Step 2 says to substitute the result into the remaining equation and solve it:

$$3x + 2y = 11$$

$$3x + 2(6-2x) = 11 \qquad \text{(Substituting)}$$

$$3x + 12-4x = 11$$

$$-x + 12 = 11$$

$$x = 1$$

Step 3 says to substitute the answer obtained into the original equation to determine the value of the second variable. Since we have already solved this equation for y in terms of x, use this form of the equation:

$$y = 6-2x$$

$$y = 6-2(1) \qquad \text{(Substituting)}$$

$$y = 4$$

The common solution, therefore, is $\underline{x = 1}$ AND $\underline{y = 4}$.

The choice of which variable to solve for and in which equation to do it (Step 1) is totally optional. Solving for y in the second equation was done because it was the easiest equation to solve. Had we chosen, for example, to solve the first equation for y, then

$$y = \frac{11-3x}{2}.$$

Substituting this fractional expression into the other equation is valid, but the resulting equation in x is much messier. The substitution method is easiest if fractions can be avoided.

ELIMINATION METHOD:

The steps of this method are best shown by example. Using the same pair of equations,

$$3x + 2y = 11$$

$$2x + y = 6$$

Step 1: Multiply both sides of the second equation by -2. (Now both equations have the same number of y's, and they are opposite in sign):

$$3x + 2y = 11$$

$$-2(2x + y) = (6)(-2)$$

$$3x + 2y = 11$$

$$-4x - 2y = -12$$

Step 2: Add the two resulting equations and solve for x:

$$\begin{array}{rcl} 3x + 2y &=& 11 \\ \underline{-4x - 2y} &\underline{=}& \underline{-12} \\ -x &=& -1 \\ x &=& 1 \end{array} \qquad \text{(Adding)}$$

Step 3: Substitute the answer obtained into either of the original equations to solve for y:

$$2x + y = 6$$

$$2(1) + y = 6 \qquad \text{(Substituting)}$$

$$y = 4.$$

The common solution, therefore, is $\underline{x = 1}$ <u>AND</u> $\underline{y = 4}$.

The choice of method is clearly optional. The substitution method is generally more efficient when one equation is easy to solve for one of the unknowns, or if the coefficients of the two equations are particularly nasty.

EXAMPLE 2:

Find the common solution to the following pair of equations in two unknowns:

$$3x - 5y = 13$$

$$2x + 3y = -2$$

Solution: Solving either equation for x or y (substitution method) will result in fractions. Therefore, the elimination method will be used:

$$\begin{array}{lrrcr}
3x - 5y = 13 & \text{MULTIPLY BY 3:} & 9x - 15y & = & 39 \\
2x + 3y = -2 & \text{MULTIPLY BY 5:} & \underline{10x + 15y} & \underline{=} & \underline{-10} \\
 & \text{ADD:} & 19x & = & 29
\end{array}$$

Solve for x: $x = 29/19$

Solve for y: $2\left(\frac{29}{19}\right) + 3y = -2$

$$\frac{58}{19} + 3y = -2$$

$$58 + 57y = -38$$

$$57y = -96$$

$$y = \frac{-96}{57} = \frac{-32}{19}$$

The common solution, therefore, is <u>$x = 29/19$</u> and <u>$y = -32/19$</u>.

Sometimes, rearrangement of one or more of the equations might be necessary before a determination of approach is made. The next example illustrates:

EXAMPLE 3:

Find the common solution to the following pair of equations in two unknowns:

$$5y + 7 = -2x$$

$$\frac{x}{2} - \frac{2y}{3} = 4$$

Solution: Rearrange the first equation and multiply the second by 6:

$5y + 7 = -2x$	REARRANGING:	$2x + 5y = -7$
$\frac{x}{2} - \frac{2y}{3} = 4$	MULTIPLY BY 6:	$3x - 4y = 24$

Using the elimination method,

$$\begin{array}{lrl} 2x + 5y = -7 & \text{MULTIPLY BY 3:} & 6x + 15y = -21 \\ 3x - 4y = 24 & \text{MULTIPLY BY 2:} & \underline{-6x + 8y = -48} \\ & \text{ADD:} & 23y = -69 \end{array}$$

Solve for y: $y = -3$

Solve for x: $2x + 5(-3) = -7$

$$2x - 15 = -7$$

$$2x = 8$$

$$x = 4$$

The common solution, therefore, is <u>x = 4</u> <u>AND</u> <u>y = -3</u>.

EXERCISES:

Find the common solution to the following pairs of equations in two unknowns:

1. $x + y = 6$
   $5x + 3y = 21$

2. $y - 4 = 3x$
   $2y - 3x = 5$

3. $4x - 5y = -1$

$3x - 2y = 1$

4. $\frac{3x}{8} + \frac{2y}{3} = 1$

$\frac{3x}{4} + \frac{5y}{6} = 1$

5. $2a = \frac{b}{3} - 1$

$\frac{a}{3} - \frac{b}{2} - 4 = 0$

2.4.1 PRACTICE PROBLEMS

Solve the following equations for the unknown value and check the solutions:

1) $3x + 16 = 5x + 8$

2) $7n - 6 = 9n - 114$

3) $8x + 7 = 5x + 1$

4) $\frac{3x}{7} = 21$

5) $\frac{2x}{7} = 3.5$

6) $2.6 = \frac{5r}{3}$

7) $2(2x - 1) + 4 = 26$

8) $8 - 4(10 - 3y) = 4y$

9) $12x - (4x - 6) = 3x - (9x - 27)$

10) $\frac{x}{5} - \frac{x - 3}{10} = 0$

11) $\frac{x}{3} - \frac{x - 8}{15} = 0$

12) $\frac{5n}{3} - \frac{2n - 2}{5} = 5 - \frac{n + 9}{3}$

13) $4(3n + 2) = 6(3 - n) + 8$

14) $\frac{2}{3} - \frac{x - 5}{5} = 2 + \frac{2x + 1}{3}$

15) $6 - (x - 7) = \frac{2}{5} - \frac{x - 3}{2}$

16) $\frac{6x + 2}{3} - \frac{2x + 3}{2} = 1 - x$

Solve each of the following formulas for the letter indicated:

17) $KE = \frac{1}{2}mv^2$, for m

18) $K = \frac{a^2c}{1 - a}$, for c

19) $V = \frac{4}{3} \pi r^3$, for r^3

20) $S = \frac{1}{2}(a + 1)$, for a

21) $S = \frac{n}{2} [2a + (n-1)d]$, for a

Solve each of the following formulas by substituting the known values:

22) $R_t = \frac{R_1 R_2}{R_1 + R_2}$; when $R_1 = 1 \times 10^2$, $R_2 = 1 \times 10^2$

23) $DPM = \frac{16(\lambda)(K)}{(6.5 \times 10^{-3}) - K}$; when $\lambda = 1 \times 10^{-1}$, $K = 2.5 \times 10^{-3}$

24) $\% = \frac{(1236 - T)F}{3 \times 10^9} \times (1 \times 10^2) - .8$; when $T = 136$, $F = 3 \times 10^5$

25) $C = \frac{5}{9}(F - 32)$ for F; when $C = 20$

Solve the following pairs of equations and check the solutions:

26) $7x - 3y = -98$
$2x + y = 11$

27) $\frac{2x}{5} - \frac{3y}{2} = 5$
$\frac{x}{5} + \frac{3y}{2} = -2$

28) $3a - 4 = 2b$
$5a = 6b - 8$

29) $3x + y = 7$
$2x - 5y = -1$

30) $2x = 5y + 4$
$3x - 2y = -16$

2.5 EXPONENTS

Complicated expressions may be simplified and algebraic solutions may be found more readily by using exponential notation. In previous sections of the Mathematics Module, some basic rules were discussed and practiced in order to properly perform the fundamental operations required to solve the expressions and equations presented at that time. However, to say that an exponent merely shows how many times a base is used as a factor is incomplete and the rules are therefore expanded to make exponential notation more useful.

2.5.1 RULES FOR EXPONENTS

(1) The product of two powers having the SAME BASE is that base raised to a power equal to the SUM of the exponents. $a^m \cdot a^n = a^{m+n}$

(2) The product of two powers having DIFFERENT BASES BUT THE SAME EXPONENTS, is equal to the product of the bases raised to the COMMON exponent. $a^m \cdot b^m = (ab)^m$

(3) A power raised to a power is equal to the base raised to the PRODUCT of the exponents. $(a^m)^n = a^{mn}$

(4) The quotient of two powers having the SAME BASE is the base raised to a power equal to the DIFFERENCE of the exponents. $\frac{a^m}{a^n} = a^{m-n}$

(5) The quotient of two powers having DIFFERENT BASES BUT THE SAME EXPONENTS is equal to the quotient of the bases raised to the COMMON exponent. $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$

(6) A base with a NEGATIVE exponent may be written as one over that base raised to the corresponding positive exponent. $a^{-m} = \frac{1}{a^m}$

(7) A number (other than zero) raised to the zero power is equal to 1. $a^0 = 1$ if $a \neq 0$

(8) A number without an exponent is assumed to have the exponent 1. A number with the exponent 1 is that number. $a^1 = a$

EXAMPLE 1

(a) Find the following product if x = 3; (RULE 1)

$$(2^{3x-d})(2^{d-2x}) = 2^{(3x-d)\ +\ (d-2x)} = 2^{3x-d+d-2x} = 2^{x} = 2^{3} = 8$$

(b) Find the following product if n = 2; (RULE 2)

$$(4^{n-1})(5^{n-1}) = (4 \bullet 5)^{n-1} = 20^{n-1} = 20^{2-1} = 20^{1} = 20$$

(c) Find the product of the following powers; (RULE 2, followed by RULE 3).

$$\left(a^{3}b^{4}c\right)^{2} = \left(a^{3}\right)^{2}\left(b^{4}\right)^{2}\left(c^{1}\right)^{2} = a^{(3\bullet 2)}b^{(4\bullet 2)}c^{(1\bullet 2)} = a^{6}b^{8}c^{2}$$

$$\left(4a^{2}\right)^{2} = \left(4^{1}\right)^{2}\left(a^{2}\right)^{2} = 4^{(1\bullet 2)}a^{(2\bullet 2)} = 4^{2}a^{4} = 16a^{4}$$

EXAMPLE 2

COMMON ERROR made when the product of two powers is indicated:

$(2)^{2}(5)^{3} \neq (2 \bullet 5)^{6}$ since,

$(2 \bullet 5)^{6} = 10^{6} = 1{,}000{,}000$ and,

$(2)^{2}(5)^{3} = (4)(125) = 500$

EXAMPLE 3

(a) Find the following quotient if a = 2; (RULE 4)

$$\frac{7^{2a+2}}{7^{a+2}} = 7^{(2a+2)\ -\ (a+2)} = 7^{2a+2-a-2} = 7^{a} = 7^{2} = 49$$

(b) Find the following quotient; (RULE 5)

$$\frac{(21)^{2}}{(7)^{2}} = \left(\frac{21}{7}\right)^{2} = (3)^{2} = 9$$

EXAMPLE 4

COMMON ERROR made when the quotient of two powers is indicated:

$$\frac{(2)^4}{(4)^2} \neq \left(\frac{2}{4}\right)^{4-2} \text{ or } \left(\frac{2}{4}\right)^2 \text{ since,}$$

$$\left(\frac{2}{4}\right)^2 = \frac{4}{16} = \frac{1}{4} \text{ and,}$$

$$\frac{(2)^4}{(4)^2} = \frac{16}{16} = 1$$

EXAMPLE 5

Evaluate the following expressions; (RULE 6)

(a) $(8 - 3)^{-2} = \dfrac{1}{(8 - 3)^2} = \dfrac{1}{(5)^2} = \dfrac{1}{25}$

(b) $(a + b)^{-2} = \dfrac{1}{(a + b)^2} = \dfrac{1}{(a + b)(a + b)} = \dfrac{1}{a^2 + 2ab + b^2}$

EXAMPLE 6

COMMON ERROR made when an indicated sum or difference is raised to a power:

(a) $(8 - 3)^{-2} = \dfrac{1}{(8 - 3)^2} \neq \dfrac{1}{(8)^2 - (3)^2} = \dfrac{1}{64 - 9} = \dfrac{1}{55}$

(b) $(a + b)^{-2} = \dfrac{1}{(a + b)^2} \neq \dfrac{1}{a^2 + b^2}$

Some expressions may be simplified and put into the proper form for applying the rules shown above.

EXAMPLE 7

$$\frac{(27)^3}{(9)^5} = \frac{(3^3)^3}{(3^2)^5} = \frac{3^9}{3^{10}} = 3^{9-10} = 3^{-1} = \frac{1}{3^1} = \frac{1}{3}$$

EXERCISES

(1) Perform the indicated operations; express all answers in the form of positive exponents:

(a) $c^{-2} \bullet c^{-3}$

(b) $(x^4)^5$

(c) $3^m \bullet 2^m$

(d) $\frac{r^8}{r^5}$

(e) $(x + y)^{-2}$

(f) $\frac{x^2y^5}{x^4y^4}$

(g) $a^{x+1} \bullet a^{x-1}$

(h) $\frac{y^{3a+1}}{y^{a-1}}$

(i) $\frac{(x^{n+2}y^{n+1})^2}{x^{2n}y^2}$

(j) $\frac{1}{(x - y)^2}$

(2) Evaluate the following expressions if x = 2 and y = 3:

(a) $x^{-2}y^0$

(b) $(xy)^{-2}$

(c) $\frac{a^{x+y}b^{x+1}}{a^{2x}b^2}$

(d) $\frac{(4)^7}{(4)^7}$

(e) $\frac{(x^{2n-3}y^{n-2})^2}{x^{n-8}y^{3n-7}}$

(f) $\frac{(2)^6}{(3)^4}$

2.5.2 FRACTIONS WITH NEGATIVE EXPONENTS

A fraction that contains a negative exponent in both numerator and denominator may be simplified in one of the following ways:

EXAMPLE 8

$$\frac{b^{-2}}{b^{-5}} = b^{-2-(-5)} = b^{-2+5} = b^3 \qquad \text{(RULE 4)}$$

EXAMPLE 9

$$\frac{b^{-2}}{b^{-5}} = b^{-2} \div b^{-5} = \frac{1}{b^2} \div \frac{1}{b^5} = \frac{1}{b^2} \times \frac{b^5}{1} = \frac{b^5}{b^2} = b^{5-2} = b^3$$

ANY POWER which is a FACTOR of the numerator or denominator can be moved from the numerator to the denominator or from the denominator to the numerator by changing the sign of the exponent.

EXAMPLE 10

$$\frac{b^{-2}}{b^{-5}} = \frac{b^5}{b^2} \text{ and } \frac{x^{-m}}{x^{-n}} = \frac{x^n}{x^m} \text{ and } \frac{r^{-a}}{t^{-b}} = \frac{t^b}{r^a} \qquad \text{(RULE 6)}$$

When the numerator and denominator contain terms which are products, divide the fraction such that one factor contains all the negative exponents. Then, simplify each FACTOR and recombine.

EXAMPLE 11

$$\frac{h^2k^{-2}}{4t^{-3}} = \frac{h^2}{4}\left(\frac{k^{-2}}{t^{-3}}\right) = \frac{h^2}{4} \bullet \frac{t^3}{k^2} = \frac{h^2t^3}{4k^2}$$

When the numerator or denominator contains the SUM of numbers with negative exponents, re-write the fraction as a complex fraction.

EXAMPLE 12

$$\frac{5 + b^{-3}}{b^{-2}} = \frac{5 + \frac{1}{b^3}}{\frac{1}{b^2}} \qquad \text{(Rewrite as complex fraction)}$$

$$= \frac{b^3}{b^3}\left(\frac{5 + \frac{1}{b^3}}{\frac{1}{b^2}}\right) = \frac{5b^3 + \frac{b^3}{b^3}}{\frac{b^3}{b^2}} \qquad \text{(Multiply by common denominator written as a fractional symbol = 1)}$$

$$= \frac{5b^3 + 1}{b} \qquad \text{(Reduce to simplest terms)}$$

EXERCISES

(1) Simplify each expression; express answers using positive exponents only.

(a) $\dfrac{x^{-2}y^{3}}{x^{3}y^{-2}}$

(b) $\dfrac{2^{-1}a^{-1}}{3a^{2}b^{-1}}$

(c) $x^{-2} - y^{-2}$

(d) $\dfrac{(2a)^{-3}b^{2}}{cd^{-2}}$

(e) $\dfrac{x^{2}y^{-3}z^{-1}}{a^{-2}bc^{-5}}$

(f) $\dfrac{a^{0}b^{2}c^{3}}{b^{-2}c^{-4}}$

(2) Evaluate the following:

(a) $\dfrac{10^{-3}}{10^{8}}$

(b) $\dfrac{10^{-3}}{10^{-5}}$

(c) $\dfrac{10^{21}}{10^{-7}}$

(d) $\dfrac{3 \times 10^{4}}{1.5 \times 10^{-1}}$

(e) $\dfrac{1}{10^{-3}} \bullet \dfrac{1}{10^{3}}$

(f) $\dfrac{1}{10^{-5}} + (2 \times 10^{5})$

2.5.3 ROOTS OF POWERS AND FRACTIONAL EXPONENTS

A root or a number is one of the EQUAL FACTORS of that number. A root of a power is one of the equal factors of that power. To indicate the root of a number or the root of a power using exponential notation, a fraction is used in which the denominator represents the ROOT INDEX (number of equal factors) and the numerator represents the power.

EXAMPLE 13

(a) Since $\sqrt[3]{16} \cdot \sqrt[3]{16} \cdot \sqrt[3]{16} = 16$, then $\sqrt[3]{16}$ is one of the equal factors of the number 16 or the power 16^{1}. Using exponents, $\sqrt[3]{16} = 16^{1/3}$.

To raise a power to a power, multiply the exponents (Rule 3). Since finding the root of a power is the inverse of finding a power, then DIVIDE THE EXPONENT BY THE ROOT INDEX.

EXAMPLE 14

(a) To raise a^2 to the third power, $(a^2)^3 = a^{2 \cdot 3} = a^6$

(b) The cube root of a^6 is $(a^6)^{\frac{1}{3}} = a^{\frac{6}{3}} = a^2$

(c) The square root of 2^3 is $(2^3)^{\frac{1}{2}} = 2^{\frac{3}{2}}$

Often two or more factors of a term are raised to the same power such as x^3y^3. Using the fundamental rules of exponents, x^3y^3 may be written as $(xy)^3$. The square root of $(xy)^3$ is indicated by $(xy)^{3/2}$.

> If a, m and n represent any numbers and x represents any number except zero, then
>
> $$(a^m)^n = a^{mn} \text{ and } \sqrt[x]{a^m} = (\sqrt[x]{a})^m = a^{\frac{m}{x}}$$

EXERCISES

Perform the indicated operations and simplify.

(1) $(x^2)^4$

(2) $(a^{-3})^2$

(3) $(a^m)^{-3}$

(4) $\sqrt[3]{x^{12}}$

(5) $\sqrt{a^{2m}}$

(6) $\sqrt{9r^{-6}}$

(7) $\sqrt[4]{16n^{16}}$

(8) $(\sqrt[6]{2})^6$

Be careful to differentiate between the minus sign that refers to the entire expression and the minus sign that refers to the base only.

EXAMPLE 15

(a) $-64^{\frac{2}{3}} = (-1)\ 64^{\frac{2}{3}} = (-1)\ (\sqrt[3]{64})^2 = (-1)(4)^2 = -16$

(b) $(-64)^{\frac{2}{3}} = \left(\sqrt[3]{-64}\right)^2 = (-4)^2 = +16$

(c) $-7^0 = (-1)(7^0) = (-1)(1) = -1$

(d) $(-7)^0 = +1$

If the fractional exponent is negative, express as one over the base raised to the corresponding positive exponent FIRST.

EXAMPLE 16

$$8^{\frac{-2}{3}} = \frac{1}{8^{2/3}} = \frac{1}{\left(\sqrt[3]{8}\right)^2} = \frac{1}{(2)^2} = \frac{1}{4}$$

EXERCISES

(1) Write each of the following in radical form.

(a) $a^{\frac{2}{3}}$ (b) $c^{0.5}$ (c) $2x^{\frac{2}{5}}$

(2) Write each of the following with fractional exponents.

(a) $\sqrt[3]{xy^2}$ (b) $4\sqrt[3]{y^2}$ (c) $\sqrt[3]{5b^2}$

(3) Evaluate the following, reduce to lowest terms.

(a) $81^{\frac{1}{4}}$

(b) $4^{-\frac{3}{2}}$

(c) $\left(\frac{-1}{8}\right)^{\frac{2}{3}}$

2.6 RADICAL EQUATIONS

If the unknown in an equation is in the radicand, the equation is called a RADICAL EQUATION. Since a radicand may be written with a fractional exponent, radical equations may be solved by applying the fundamental rules of exponents.

EXAMPLE 1

Find the solution to the equation $2\left(\sqrt[3]{x^2}\right) + 1 = 9$

(a) $2x^{\frac{2}{3}} + 1 = 9$ (Rewrite the equation using exponents and isolate the unknown to one side of the equation.)

$2x^{\frac{2}{3}} = 8$

$x^{\frac{2}{3}} = 4$

Examine this equation for a moment. Since $x^{\frac{2}{3}} = (x^2)^{\frac{1}{3}}$, it is clear that if x is either positive or negative, x^2 is always positive, so $x^{2/3}$ is always positive. Therefore, when the numerator of a fractional exponent is even, there will be two solutions to the equation, equal in algebraic value but opposite in sign. (If the numerator of the fractional exponent is odd, there will be only one solution).

(b) $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = \pm\, 4^{\frac{3}{2}}$ (Raise BOTH SIDES of the equation to a power that will make the exponent of the unknown equal to 1. The ± is placed in front of $4^{3/2}$ so that both solutions can be obtained.)

$x^1 = \pm\, 4^{\frac{3}{2}}$

(c) $x = \pm\left(\sqrt{4}\right)^3$

$x = \pm\, 2^3$ (Solve the equation for the value of x.)

$x = \pm 8$ Answer.

(d) Check the root in the original equation.

$2\sqrt[3]{x^2} + 1 = 9$

$(2)(\sqrt[3]{8^2}) + 1 = 9$ or $(2)(\sqrt[3]{(-8)^2}) + 1 = 9$

continued...

$$(2)(\sqrt[3]{64}) + 1 = 9$$

$$(2 \cdot 4) + 1 = 9$$

$$8 + 1 = 9$$

$$9 = 9$$

2.7 EXPONENTIAL EQUATION

An equation in which the unknown is an exponent is called an exponential equation. An exponential equation may be solved by:

(a) Isolate the term containing the unknown on one side of the equation,

(b) Express both sides of the equation as powers of the unknown's base and equate the exponents.

EXAMPLE 1

Find the value of x in the equation $3^{x+2} - 5 = 76$

(a) $3^{x+2} - 5 = 76$ (Isolate the unknown on one side)

$3^{x+2} = 81$

(b) $3^{x+2} = 3^4$ (Express both sides as powers of the exponent's base.)

(c) $x + 2 = 4$ (Equate the exponents)

$x = 2$ (Answer, or root)

(d) Check by substituting in the original equation.

$$3^{x+2} - 5 = 76$$

$$3^{2+2} - 5 = 76$$

$$3^4 - 5 = 76$$

$$81 - 5 = 76$$

$$76 = 76$$

Exponential equations in which the other member is not an integral power of the unknown's base will be discussed in another segment of the Mathematics Module.

EXERCISES

Solve for x and check your answer.

(1) $x^{\frac{2}{3}} - 1 = 3$

(2) $2^{x+1} = 32$

(3) $3^{1-x} - 3 = 24$

2.7.1 PRACTICE PROBLEMS

Simplify the following expressions

1) $a^2 \cdot a^{3n}$

2) $(2^3)^2$

3) $\dfrac{18x^3y^4}{54xy^5}$

4) $\dfrac{(5)^9}{(5)^9}$

5) $\dfrac{(x^2)^3y^7}{(xy)^5}$

6) $x^2\left(\dfrac{x}{y}\right)^3$

7) $\dfrac{(3a^2)^0}{x^{-2}y^2}$

8) $\dfrac{a^{-3}b^2c}{ab^{-1}c^{-3}}$

9) $\dfrac{a^{3k}x^{k+1}}{a^kx^2}$

10) $\dfrac{(x^{n-2}y^{n+1})^4}{(x^{n+1}y^{n-1})^3}$

11) $\left(\dfrac{x^{-1}y^3}{2x^0y^{-5}}\right)^{-2}$

12) x^0y^{-4}

Evaluate the following if $x = 2$.

13) $(x^{-3})^2$

14) $\sqrt[3]{x^{12}}$

15) $\sqrt{x^{-2}}$

16) $x^{\frac{3}{2}}$

Perform the indicated operations and simplify each result.

17) $27^{-2/3} + 2(10)^0 - \left(\dfrac{1}{8^{-1/3}}\right)$

18) $\left(\dfrac{1}{4}\right)^{\frac{-3}{2}} - 27^{\frac{1}{3}} + 1^{\frac{4}{5}}$

19) $(5m^{1/2}n^{-1/2})(4m^{-3/4}n^{1/4})$

Solve for x and check your answers.

20) $x^{\frac{3}{2}} = 27$

21) $x^{\frac{2}{3}} - 1 = 15$

22) $4x^{-\frac{1}{3}} = 12$

Solve for x and check your answers.

23) $2^{x+1} = 32$

24) $4^{x-2} = 64$

25) $2^{x+1} = \frac{1}{32}$

Evaluate the following; express answers using scientific notation.

26) $(2 \times 10^3)^2(1 \times 10^4)^{-2}$

27) $(1{,}000{,}000)^3(.0001)^3$

28) $\dfrac{(5 \times 10^8)^2}{(1 \times 10^2)^8}$

29) $\dfrac{(1 \times 10^{-2})^4}{(1 \times 10^2)^{-4}}$

30) $\sqrt{1 \times 10^4} \bullet \sqrt{1 \times 10^{12}}$

31) $\sqrt[3]{8 \times 10^{15}} \div \sqrt[3]{1 \times 10^3}$

2.8 FACTORING ALGEBRAIC EXPRESSIONS

2.8.1 INTRODUCTION

An important process for simplifying complex algebraic expressions is referred to as FACTORING. Because factoring consists of "undoing" multiplication, success in the process is closely related to skill in multiplying.

TO FACTOR A NUMBER IS TO EXPRESS THE NUMBER AS A PRODUCT OF TWO OR MORE OF ITS COMPONENTS.

If required to solve polynomial equations such as $x^3 + 3x^2 - 9x - 27 = 0$, the polynomial must first be factored. Similiarly, in order to simplify polynomial fractions such as $\frac{6x^2 - 7x - 20}{2x^2 + 9x + 10}$, the numerator and denominator must first be factored. This portion of the Mathematics Module is used to acquaint the student with various techniques that may be used to factor algebraic expressions.

2.8.2 PRIME FACTORS

Some numbers are referred to as being prime. A prime number has no factors except itself and one. Thus, 3 is prime, since its only factors are 3 and 1.

Likewise, the binomial expression $x^2 + y^2$ is prime, for its only factors are $x^2 + y^2$ and 1.

However, many numbers are not prime. The factors of 6 are not only 6 and 1, but also 2 and 3. And the expression $x^2 - y^2$ not only equals $1\,(x^2 - y^2)$, it also equals $(x+y)(x-y)$.

Every product can be expressed by means of prime factors. FACTORING IS THE PROCESS OF IDENTIFYING THESE FACTORS.

A PRIME FACTOR IS A POSITIVE NUMBER p, WHERE $p \neq 1$, WHICH HAS ONLY TWO POSITIVE INTEGRAL COMPONENTS: 1 and p.

EXAMPLE 1

Express the number 12 in terms of prime factors.

$$\begin{aligned} 12 &= (6)(2) \\ &= (3)(2)(2) \end{aligned}$$

Thus, the PRIME FACTORIZATION OF 12 is $(3)(2)^2$.

EXAMPLE 2

Express the number 126 in terms of prime factors.

$$\begin{aligned} 126 &= (2)(63) \\ &= (2)(3)(21) \\ &= (2)(3)(3)(7) \\ &= (2)(3)^2(7) \end{aligned}$$

Thus, the prime factorization of 126 is $(2)(3)^2(7)$.

EXERCISES

Write the following numbers in terms of their prime factors.

(1) 35	(4) 49
(2) 36	(5) 100
(3) 42	(6) 144

2.8.3 COMMON FACTORS

Whenever any expression is to be factored, look first for a factor that is common to each term; then, to find the other factor, divide each term in the expression by this common factor.

Consider the expression $5a + 5b - 10c$. It is clear that the factor 5 contained in each term of the expression; to factor this product or to "undo the multiplication", divide each term by the common factor 5.

$$5a + 5b - 10c = 5(a + b - 2c)$$

There may be more than one such factor; for example $4a$ is common to each term in the expression $8a^2 + 4a$. So divide each term by $4a$ to determine the other factor;

$$8a^2 + 4a = 4a(2a + 1)$$

It should be noted that when factoring, monomial factors need not be written as the product of prime numbers; however, polynomial expressions should be reduced to their prime factors.

One use of the factoring process is to help solve all literal equations as shown in the following example.

EXAMPLE 3

Solve the following equation for x.

$$ax - bx = a^2 - ab$$

$$x(a - b) = a(a-b) \qquad \text{(Find common factors)}$$

$$x = a \qquad \text{(Divide each side by } a - b\text{)}$$

EXERCISES

Write the factors of the following expressions.

1) $2x - 2y$

2) $ax + bx$

3) $ax^2 + 3x^3a$

4) $12m^3 - 6m$

5) $2\pi rh + 2\pi r^2$

6) $.6xy - .8xy^2$

7) $2xyz + xy + 5yz$

8) $3x^2y + 6y^2x$

Solve each equation for the letter indicated.

9) $ax + bx = 2a + 2b$; for x

10) $my - y = m$; for y

11) $ax - 2 = a - 2x$; for x

12) $m(r - p) = n(p - r)$; for r

13) $an + bn = 2a$; for n

14) $xy + 2 = x + 2y$; for x

15) $b(1 - n) = a - an$; for n

16) $x(a - b) = y(b - a)$; for a

2.8.4 FACTORING THE DIFFERENCE OF TWO SQUARES

Consider the following multiplication examples:

$$(3x + 2)(3x - 2) = 9x^2 - 4$$

$$(a + 3b)(a - 3b) = a^2 - 9b^2$$

$$(4x + yz)(4x - yz) = 16x^2 - y^2z^2$$

Notice that each product resulted in a two-term polynomial. Moreover, each term is a <u>perfect square</u>:

$$9x^2 = (3x)^2 , \quad 4 = (2)^2$$

$$a^2 = (a)^2 , \quad 9b^2 = (3b)^2$$

$$16x^2 = (4x)^2 , \quad y^2z^2 = (yz)^2$$

Whenever two perfect squares are separated by a minus sign, the expression is called the difference of two squares. The multiplication examples above show how to <u>factor</u> the difference of two squares, since factoring is just the inverse operation of multiplication. In general,

$$a^2 - b^2 = (a + b)(a - b)$$

<u>EXAMPLES</u>

$$25x^2 - 9 = (5x + 3)(5x - 3)$$

$$a^2 - 81 = (a + 9)(a - 9)$$

$$5x^2 - 20y^2 = 5(x^2 - 4y^2) = 5(x + 2y)(x - 2y)$$

EXERCISES

Factor the following expressions.

1) $x^2 - y^2$

2) $144 - y^2$

3) $4a^2x^2 - 9b^2y^2$

4) $-49x^2 + 64y^2$

5) $27a^2x^2 - 3$

6) $(a + b)^2 - (c + d)^2$

7) $x^2 - 25$

8) $3x^2 - 12y^2$

2.9 ALGEBRAIC FRACTIONS/PHYSICAL UNITS:

In Chapter 1, the rules for performing the basic operations on arithmetic fractions were discussed. When dealing with algebraic fractions, that is, fractions containing literal factors and/or terms, the same rules apply:

1) $\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$

2) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \bullet \frac{d}{c} = \frac{ad}{bc}$

3) Algebraic fractions cannot be combined (added or subtracted) unless they have the same denominator.

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

EXAMPLES:

Arithmetic: / Algebra:

$$\frac{2}{9} \bullet \frac{3}{4} = \frac{\overset{1}{\cancel{2}}}{3 \bullet \underset{1}{\cancel{3}}} \bullet \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}} \bullet 2} = \frac{1}{6} \quad \longleftrightarrow \quad \frac{x^2}{yz} \bullet \frac{z^2}{xy} = \frac{\overset{1}{\cancel{x}} \bullet x}{y \bullet \underset{1}{\cancel{z}}} \bullet \frac{\overset{1}{\cancel{z}} \bullet z}{\underset{1}{\cancel{x}} \bullet y} = \frac{xz}{y^2}$$

$$\frac{25}{36} \div \frac{5}{9} = \frac{\overset{5}{\cancel{25}}}{\underset{4}{\cancel{36}}} \bullet \frac{\overset{1}{\cancel{9}}}{\underset{1}{\cancel{5}}} = 5/4 \quad \longleftrightarrow \quad \frac{3a^2b}{c^3d} \div \frac{a}{cd} = \frac{3\overset{a}{\cancel{a^2}}b}{\underset{c^2 \bullet 1}{\cancel{c^3}\cancel{d}}} \bullet \frac{\overset{1 \bullet 1}{\cancel{cd}}}{\underset{1}{\cancel{a}}} = \frac{3ab}{c^2}$$

$$\frac{2}{11} + \frac{5}{11} = \frac{2+5}{11} = \frac{7}{11} \quad \longleftrightarrow \quad \frac{a^2}{3x} + \frac{2b}{3x} = \frac{a^2 + 2b}{3x}$$

Algebraic fractions in the Fundamentals courses which follow deal mainly with <u>physical units</u>:

$$\frac{ft}{sec^{2}} \bullet \frac{lbf}{ft} \div \frac{lbf}{sec} = \frac{\overset{1}{\cancel{ft}}}{\underset{sec}{\cancel{sec^{2}}}} \bullet \frac{\overset{1}{\cancel{lbf}}}{\underset{1}{\cancel{ft}}} \bullet \frac{\overset{1}{\cancel{sec}}}{\underset{1}{\cancel{lbf}}} = \frac{1}{sec}, \text{ or } sec^{-1}$$

$$\frac{lbf}{in^{2}} \bullet \frac{ft^{3}}{lbm} \bullet \left(\frac{in}{ft}\right)^{2} = \frac{lbf}{\underset{1}{\cancel{in^{2}}}} \bullet \frac{\overset{ft}{\cancel{ft^{3}}}}{lbm} \bullet \frac{\overset{1}{\cancel{in^{2}}}}{\underset{1}{\cancel{ft^{2}}}} = \frac{ft \bullet lbf}{lbm}$$

$$\frac{lbm\left(\frac{ft}{sec}\right)^{2}\left(\frac{BTU}{ft \bullet lbf}\right)}{\frac{ft \bullet lbm}{lbf \bullet sec^{2}}} = \frac{\overset{1}{\cancel{lbm}}}{1} \bullet \frac{\overset{\overset{1}{\cancel{ft}}}{\cancel{ft^{2}}}}{\underset{1}{\cancel{sec^{2}}}} \bullet \frac{BTU}{\underset{1}{\cancel{ft}} \bullet \underset{1}{\cancel{lbf}}} \bullet \frac{\overset{1}{\cancel{lbf}} \bullet \overset{1}{\cancel{sec}}}{\underset{1}{\cancel{ft}} \bullet \underset{1}{\cancel{lbm}}} = BTU$$

$$\frac{gal}{min} \bullet \frac{lbf}{in^{2}} \bullet \frac{ft^{3}}{gal} \bullet \frac{in^{2}}{ft^{2}} \bullet \frac{min}{sec} \bullet \frac{hp}{\frac{ft \bullet lbf}{sec}} =$$

$$\frac{gal}{min} \bullet \frac{lbf}{in^{2}} \bullet \frac{ft^{3}}{gal} \bullet \frac{in^{2}}{ft^{2}} \bullet \frac{min}{sec} \bullet \frac{hp}{1} \bullet \frac{sec}{ft \bullet lbf} = hp$$

Just as algebraic fractions cannot be combined unless they have identical denominators, algebraic terms with units cannot be combined unless their units are identical.

$$12cm + 3in = ?$$

Throughout the courses which follow, you will be given tables which provide <u>conversion factors</u>. For instance,

$$1in = 2.54cm \quad \text{means} \quad \frac{1in}{2.54cm} = \frac{2.54cm}{1in} = 1.$$

Therefore, if you prefer the answer to 12cm + 3in in terms on <u>inches</u>,

$$12\text{cm} + 3\text{in} = 12\text{cm}\left(\frac{1\text{in}}{2.54\text{cm}}\right) + 3\text{in} = 4.72\text{in} + 3\text{in} = 7.72\text{in},\ \underline{\text{answer}}.$$

If the answer is desired in <u>centimeters</u>,

$$12\text{cm} + 3\text{in} = 12\text{cm} + 3\text{in}\left(\frac{2.54\text{cm}}{\text{in}}\right) = 12\text{cm} + 7.62\text{cm} = 19.62\text{cm},\ \underline{\text{answer}}.$$

Other examples of using conversion factors:

<u>EXAMPLE 1</u>

Convert 60 $\frac{\text{mi}}{\text{hr}}$ to $\frac{\text{ft}}{\text{sec}}$.

Solution: Since 1mi = 5280 ft, and 1 hr = 60 min, 1 min = 60 sec,

$$\left(60\ \frac{\text{mi}}{\text{hr}}\right)\left(\frac{5280\text{ft}}{\text{mi}}\right)\left(\frac{1\text{ hr}}{60\text{ min}}\right)\left(\frac{1\text{ min}}{60\text{ sec}}\right) = 88\text{ ft/sec}.$$

<u>EXAMPLE 2</u>

$$\frac{\left(10\ \frac{\text{ft}}{\text{sec}}\right)^2}{64.4\ \frac{\text{ft}\bullet\text{lbm}}{\text{lbf}\bullet\text{sec}^2}} + \frac{3\text{ft}(32.2\text{ft/sec}^2)}{32.2\ \frac{\text{ft}\bullet\text{lbm}}{\text{lbf}\bullet\text{sec}^2}} + (40\text{ lbf/in}^2)(0.16\text{ ft}^3/\text{lbm}) = ?$$

$$= \frac{10^2}{64.4}\ \frac{\text{ft}^2}{\text{sec}^2}\bullet\frac{\text{lbf}\bullet\text{sec}^2}{\text{ft}\bullet\text{lbm}} + \frac{3(32.2)}{32.2}\ \frac{\text{ft}}{1}\bullet\frac{\text{ft}}{\text{sec}^2}\bullet\frac{\text{lbf}\bullet\text{sec}^2}{\text{ft}\bullet\text{lbm}} + (40)(0.16)\ \frac{\text{lbf}}{\text{in}^2}\bullet\frac{\text{ft}^3}{\text{lbm}}$$

The three terms <u>cannot</u> <u>be</u> <u>combined</u> unless they are <u>unit</u> <u>consistent</u>.
Notice that the first two terms are consistent:

$$= \overbrace{1.55\ \frac{ft \bullet lbf}{lbm} + 3\ \frac{ft \bullet lbf}{lbm}}^{\text{Same Units}} + 6.4\ \frac{lbf}{in^2} \bullet \frac{ft^3}{lbm}$$

Since $1\ ft^2 = 144\ in^2$,

$$= 1.55\ \frac{ft \bullet lbf}{lbm} + 3\ \frac{ft \bullet lbf}{lbm} + 6.4\ \frac{lbf}{in^2} \bullet \frac{ft^3}{lbm}\left(\frac{144 in^2}{ft^2}\right)$$

$$= 1.55\ \frac{ft \bullet lbf}{lbm} + 3\ \frac{ft \bullet lbf}{lbm} + 921.6\ \frac{ft \bullet lbf}{lbm}$$

$$= 926.15\ \frac{ftlbf}{lbm},\ \underline{\text{answer}}.$$

The problems which follow involve formula evaluations where the variables contain units. In some cases, simple fraction manipulation is all that is required to obtain the desired unit. In other situations conversion factors may need to be introduced.

EXERCISES

1. Given that flow energy (FE) is equal to the product of fluid pressure (P) and fluid specific volume (ν), i.e., FE = Pν, determine FE

   (in units of ft•lbf/lbm) if P = 120 lbf/in^2 and $\nu = 0.01630\ ft^3/lbm$.

2. Given that $\dot{Q} = \rho\dot{V}c\ (T_2 - T_1)$, determine $\dot{Q}$ (in Mw)

   if $\rho = 44.9\ lbm/ft^3$, $\dot{V} = 4.45 \times 10^5$ gal/min., c = 1.42 BTU/lbm.°F,

   $T_2 = 621°F$, $T_1 = 564°F$. 1 Mw = 3.413×10^6 BTU/hr and

   $1\ ft^3 = 7.4805$ gal.

3. Given that $KE = \frac{mv^2}{2g_c}$, determine KE (in BTU's) given that m = 5 lbm,

$v = 15$ ft/sec, $g_c = 32.2$ ft.lbm/lbf • sec^2, and 1 BTU = 778 ft.lbf

4. Given that $2as = v^2 - v_0^2$, determine a (in ft/sec^2) if

$s = 82$ ft, $v_0 = 22$ mi/hr, $v = 28$ mi/hr, and 1 mi = 5280 ft.

5. Given that $P_2 = P_1 + \frac{\rho g h}{g_c}$, determine P_2 (in lbf/in^2) if

$P_1 = 10$ lbf/in^2, $\rho = 62.4$ lbm/ft^3, $g_c = 32.2$ ft.lbm/lbf . sec^2,

and $h = 33.9$ ft, $g = 32.2$ ft/sec^2

6. Given that $Pwr = \dot{V}\Delta P$, determine Pwr (in hp) given that

$\dot{V} = 400$ gal/min, $\Delta P = 45$ lbf/in^2, 1 hp = 550 ft.lbf/sec, 7.4805 gal = 1 ft^3.

7. Given the formula

$$\frac{V_1^2}{2g_c} + \frac{h_1 g}{g_c} + P_1 \nu = \frac{V_2^2}{2g_c} + \frac{h_2 g}{g_c} + P_2 \nu$$

Determine P_2 (in lbf/in^2) if $V_1 = 8$ ft/sec

$V_2 = 10$ ft/sec

$h_1 = 3$ ft

$h_2 = 3$ ft

$P_1 = 20\ lbf/in^2$

$\nu = 0.016\ ft^3/lbm$

$g = 32.2\ ft/sec^2$

$g_c = 32.2\ ft.lbm/lbf.sec^2$

2.10 HIGHER DEGREE EQUATIONS

Equations containing unknown values that are multiplied by other unknowns or by themselves are called higher degree equations. If the highest power of the unknown in an equation is the SQUARE, it is a second degree equation; if the highest power is a CUBE, it is a third degree equation; and so on.

Second degree equations are also commonly called QUADRATIC EQUATIONS and are the type we will discuss in this portion of the Mathematics Module. The following example shows the standard form of a quadratic equation.

EXAMPLE 1

The standard quadratic form is

$$ax^2 + bx + c = 0$$

where the letter a represents any constant except zero, b and c represents any constants (which may be zero), and x represents the unknown variable.

The fundamental rules used to solve linear equations are applicable to higher degree equations. A solution, or root, of an equation is still a replacement for the VARIABLE which will make the algebraic sentence a true statement. However, where the solution of linear equations resulted in a single value for each unknown, the solution of a quadratic equation results in two values for the unknown.

2.10.1 SOLVING EQUATIONS USING THE QUADRATIC FORMULA

The quadratic formula may be used to solve any quadratic equation in one unknown. To derive the formula is beyond the scope of this Mathematics Module, but the quadratic formula is presented so the students will have a means with which to solve those equations.

EXAMPLE 2

The quadratic formula

$$\boxed{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

where the literal numbers are represented in the general quadratic trinomial $ax^2 + bx + c$ in which $\underline{a}$ represents any number except zero, $\underline{b}$ and $\underline{c}$ represent any numbers including zero.

TO USE THE QUADRATIC FORMULA, FIRST ARRANGE THE TERMS OF THE EQUATION TO FIT THE STANDARD QUADRATIC FORM. Then, substitute the numerical coefficients of the equation into the formula and solve for the unknown value.

EXAMPLE 3

Solve $4x^2 + x = 3$

$4x^2 + x - 3 = 0$ (Rearrange terms to standard form)

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(4)(-3)}}{2(4)}$$

(Substitute the corresponding values of a, b and c. Set equal to the unknown, x)

$$x = \frac{-1 \pm \sqrt{1 + 48}}{8} = \frac{-1 \pm \sqrt{49}}{8}$$

$$x = \frac{-1 \pm 7}{8}$$

(Solve for the two possible values of x)

$x = \frac{-1 + 7}{8}$	$x = \frac{-1 - 7}{8}$
$x = \frac{6}{8}$	$x = \frac{-8}{8}$
$x = \frac{3}{4}$	$x = -1$

Each possible solution is checked by substituting the value of x into the ORIGINAL equation;

$4x^2 + x = 3$	$x^2 + x = 3$
$4(3/4)^2 + 3/4 = 3$	$4(-1)^2 + (-1) = 3$
$4(9/16) + 3/4 = 3$	$4(1) + (-1) = 3$
$\frac{36 + 12}{16} = 3$	$3 = 3$
$3 = 3$	

Thus, the solution set to $4x^2 + x = 3$ is $\left\{\frac{3}{4}, -1\right\}$

EXAMPLE 4

Solve for x in the equation $x^2 - 4x + 4 = 0$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16 - 16}}{2}$$

$$x = \frac{4 \pm \sqrt{0}}{2}$$

$x = 2 \pm 0$

The solution set is $\{2,2\}$ and is referred to as a DOUBLE ROOT.

A quadratic equation may also produce irrational roots, as shown in the next example. When the roots are irrational, use your calculator to determine the approximate solutions (see Example 5).

EXAMPLE 5

Solve $2x^2 - 6x + 3 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(3)}}{4}$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{4}$$

$$x = \frac{6 \pm \sqrt{12}}{4} \approx \frac{6 \pm 3.464}{4}$$

$x \quad 2.366,\ 0.641$

Thus, the solution set is $\{2.366\ ,\ 0.641\}$

EXERCISES

Solve each of the following:

a) $x^2 - 2x - 3 = 0$

b) $x^2 - 9 = 0$

c) $x^2 - 4x = 0$

d) $x^2 - 7x + 12 = 0$

e) $-6x^2 + x + 1 = 0$

f) $4x^2 = 6x$

2.10.2 PRACTICE PROBLEMS

Solve each of the following equations using the quadratic formula.

1) $x^2 - 4x + 3 = 0$

2) $r^2 - 4r - 32 = 0$

3) $x^2 - 9x = 0$

4) $16y^2 + 1 = 8y$

5) $3x^2 - 2x - 5 = 0$

6) $5 - x - x^2 = 0$

7) $(x+2)(x-3) = 2x - 1$

8) $x^2 = 8x + 4$

9) $25x^2 + 10x + 1 = 0$

10) $2x^2 - 7x = 15$

11) $2x^2 + 4x - 3 = 0$

12) $t^2 = \frac{1}{2}t - \frac{1}{16}$

13) $3y^2 = -3y - 1$

14) $x^2 + 9x + 18 = 0$

Chapter Two: SOLUTIONS to EXERCISES

Page 4, 5 :

1) a) $14x - 7$ b) $4y + 8$ c) $a + 4b$ d) $2m + 6n$ e) $3x^2 + 4x - 15$

f) $6b^3 + 2$ g) $2a^3 + 2b^2 + b - 4$ h) $1.9m - 1.9n$ i) $22(a+b) + 5ab$

j) $9x^2y + 2$ k) $3x^2 + 5x$

2) a) 4 b) 3 c) 12 d) 18 e) 15 f) -7 g) 4 h) 1

i) 37 j) 63 k) 20 l) 10

Page 7 :

1) $8x^3 - 6x^2 - 2x + 19$ 2) $15a^4 - 7a^3 + 8a^2 - 1$ 3) $x^3y - 2x^2y^2 - 5xy^3$

4) $13b^4 - a^3b^2$ 5) $17n^3 + 3n^2 - 4n - 11$ 6) $13(a+b)^2 + 1$

7) $2x^3 - 7x^2 - 11$ 8) $9a^4 - 12a^3 + 6a^2 + a - 9$

9) $x^4 + 3x^3 - 3x^2 + 8x + 4$ 10) $a^4 - 2a^2 - 10a + 11$

11) $(x+y)^2 - (x+y) + 1$ 12) $(n-b)^2 + 3(n-b) - 1$

Page 8 :

1) a^5 2) x^6 3) x^{a+b} 4) y^{2x} 5) $24x^6$ 6) $56x^3y^2$ 7) $15a^3b^3$

8) $-27a^3b^4c^2$ 9) $-36r^3s^3t^3$ 10) $-24a^6$ 11) $60x^2y^4$ 12) $2y^2$

Page 9 :

1) $3a^4 - 2a^3 + 5a^2$ 2) $-3a^4 + 2a^3 - 5a^2$ 3) $-9n^3 + 3n^2 - 6n$

4) $48x^5 - 12x^4 + 20x^3 - 4x^2$ 5) $2n^3 + n^2 - \frac{1}{2}n$ 6) $a^3 - 10a^2 - 21a$

Page 10 :

1) $6x^2 + 5x - 56$ 2) $20c^2 + 9c - 20$ 3) $48a^2 - 22ab + 2b^2$

4) $m^3 - mn^2 - m^2n + n^3$ 5) $6x^2 + 13x + 6$ 6) $x^2 - y^2$

Page 11:

1) $x^2 + 2xy + y^2$ 2) $x^2 - 2xy + y^2$ 3) $x^3 - 3x^2y + 3xy^2 - y^3$

4) $x^3 + y^3$ 5) $x^3 - y^3$ 6) $9x^2 + 6x + 1$ 7) $x^4 - 2x^2y + y^2$

8) $r^3 - 3r^2 + 3r - 1$ 9) $x^4 + 3x^3 - x^2 - 2x + 1$ 10) $x^4 - 2x^2 + 1$

Page 12:

1) $2a$ 2) $5r$ 3) $-4x^3$ 4) $-2b^2$ 5) $3xy^2$ 6) $6c$

Page 13:

1) $5x^2 + 3x - 6$ 2) $3a^3 - 2a^2 + a$ 3) $-2x^2 + 3xy + 8y^2$

4) $-6a^2 - 3ab + 4b^2$ 5) $-r + rs + 5s$ 6) $4a - 3 + \frac{1}{a}$

Page 21-23:

1) $-\frac{4}{3}$ 2) 4 3) $\frac{29}{6}$ 4) 36 5) $\frac{6}{7}$

6) $-\frac{21}{2}$ 7) 10 8) 15 9) 6 10) $\frac{4}{3}$

Page 26:

1) $r = \frac{c}{2\pi}$ 2) $F = \frac{9}{5}C + 32$ 3) $h = \frac{3V}{\pi r^2}$ 4) $T_2 = \frac{P_2V_2T_1}{P_1V_1}$

Page 31,32:

1) $x = \frac{3}{2}, y = \frac{9}{2}$ 2) $x = -1, y = 1$ 3) $x = 1, y = 1$

4) $x = -\frac{8}{9}, y = 2$ 5) $a = -\frac{33}{16}, b = -\frac{75}{8}$

Page 38 :

1) a) $\frac{1}{c^5}$ b) x^{20} c) 6^m d) r^3 e) $\frac{1}{x^2 + 2xy + y^2}$

f) $\frac{y}{x^2}$ g) a^{2x} h) y^{2a+2} i) x^4y^{2n} j) $\frac{1}{x^2 - 2xy + y^2}$

2) a) $\frac{1}{4}$ b) $\frac{1}{36}$ c) ab d) 1 e) $2^{3n+2} \cdot 3^{3-n}$ f) $\frac{64}{81}$

Page 40 :

1) a) $\frac{y^5}{x^5}$ b) $\frac{b}{6a^3}$ c) $\frac{1}{x^2} - \frac{1}{y^2}$ d) $\frac{b^2d^2}{8a^3c}$

e) $\frac{a^2x^2c^5}{by^3z}$ f) b^4c^7

2) a) $\frac{1}{10^{11}}$ b) 10^2 c) 10^{28} d) 2×10^5 e) 1 f) 3×10^5

Page 41 :

1) x^8 2) a^{-6} 3) a^{-3m} 4) x^4 5) a^m 6) $3r^{-3}$

7) $2n^4$ 8) 2

Page 42 :

1) a) $\sqrt[3]{a^2}$ b) $\sqrt{c}$ c) $2\sqrt[5]{x^2}$

2) a) $x^{1/3}y^{2/3}$ b) $4y^{2/3}$ c) $5^{1/3}b^{2/3}$

3) a) 3 b) 1/8 c) 1/4

Page 45 :

1) ± 8 2) 4 3) -2

Page 50 :

1) $2(x-y)$ 2) $x(a+b)$ 3) $ax^2(1+3x)$ 4) $6m(2m^2-1)$

5) $2\pi r(h+r)$ 6) $0.2xy(3-4y)$ 7) $y(2xz+x+5z)$

8) $3xy(x+2y)$ 9) $x=2$ 10 $y=\frac{m}{m-1}$ 11) $x=1$

12) $r=p$ 13) $n=\frac{2a}{a+b}$ 14) $x=2$ 15) $n=1$ 16) $a=b$

Page 52 :

1) $(x-y)(x+y)$ 2) $(12-y)(12+y)$ 3) $(2ax-3by)(2ax+3by)$

4) $(8y-7x)(8y+7x)$ 5) $3(3ax+1)(3ax-1)$

6) $(a+b+c+d)(a+b-c-d)$ 7) $(x-5)(x+5)$ 8) $3(x-2y)(x+2y)$

Page 56-58:

1) $282\ \frac{ft.lbf}{lbm}$ 2) 3801 Mw 3) 2.25×10^{-2} BTU

4) $3.93\ \frac{ft}{sec^2}$ 5) $24.7\ \frac{lbf}{in^2}$ 6) 10.5 hp 7) $19.8\ \frac{lbf}{in^2}$

Page 62 :

a) $\{3, -1\}$ b) $\{3, -3\}$ c) $\{0, 4\}$ d) $\{4, 3\}$

e) $\{-\frac{1}{3}, \frac{1}{2}\}$ f) $\{0, \frac{3}{2}\}$

Chapter Two: SOLUTIONS to PRACTICE PROBLEMS

Page 14,15:

1) $4x + 2y$ 2) $2.6a + 1.4b$ 3) $\frac{3}{8}x - \frac{1}{9}y$ 4) $6a^2b + 1$

5) $5(a + b)$ 6) 7 7) 108 8) 12 9) 6 10) 32

11) 64 12) -8 13) -22 14) -39 15) 36

16) $5x - 3$ 17) $7a - 9$ 18) $-5x - 2y$ 19) $2x + 10$

20) $3x + 30$ 21) $6x - 12$ 22) $5r^3 + 14r^2 - 7r + 7$

23) $3(x + y)^3 - 1$ 24) $4ax - by$ 25) $2dn$ 26) $4x^3y - 5x^2y^2 - 5xy^3$

27) $x + a - 2b$ 28) $6x + 6y$ 29) h^9 30) $20n^5$ 31) $10x^2$

32) y^{2x} 33) $12x^3$ 34) $-12x^3y^3$ 35) $-15x^3y^4z^2$ 36) $-8a^3x^3$

37) $5x^4 - 2x^3 + 3x^2$ 38) $-4a^5 + 2a^4 + 5a^3 - a^2$ 39) $15r^4 - 10r^3 + 5r^2$

40) $\frac{1}{2}x^3 - 3x^2 - \frac{1}{4}x$ 41) $6x^2 - 13x + 6$ 42) $x^2 + ax - 6a^2$

43) $x^4 - 4$ 44) $x^2 + 8x + 16$ 45) $x^3 + 3x^2 - 9x - 27$

46) $3x^3 + 4x^2 - x - 2$ 47) $a^3 - 3a^2 + 3a - 1$

48) $2x^4 - x^3 - 2x^2 + 13x - 6$ 49) $5x^4 + 2x^3 - x^2 + x - 1$

50) $a^4 + 7a^3 + 18a^2 + 20a + 8$ 51) $2xyz$ 52) -1 53) $-\frac{1}{2}$

54) $(mn)^3$ 55) $3(ab)^2$ 56) $a^2 - a + 1$ 57) $-3x^2 + 6x + \frac{1}{x}$

58) $5(x + y)^2$ 59) $3r^2 + 2r + \frac{1}{r^3}$ 60) $-\frac{8b}{a}$

Page 33,34:

1) $x = 4$ 2) $n = 54$ 3) $x = -2$ 4) $x = 49$ 5) $x = 12.25$

6) $r = 1.56$ 7) $x = 6$ 8) $y = 4$ 9) $x = 3/2$ 10) $x = -3$

11) $x = -2$ 12) $n = 1$ 13) $n = 1$ 14) $x = \frac{-10}{13}$ 15) $x = \frac{111}{5}$

Page 33,34 (Cont'd):

16) $x = \frac{11}{12}$ 17) $m = \frac{2KE}{v^2}$ 18) $\frac{K(1 - a)}{a^2} = C$

19) $r^3 = \frac{3v}{4\pi}$ 20) $2s - 1 = a$ 21) $a = \frac{S}{n} - \frac{nd}{2} + \frac{d}{2}$

22) $R_t = 5 \times 10^1$ 23) DPM = 1 24) % = 10.2

25) F = 68 26) $x = -5,\ y = 21$ 27) $x = 5,\ y = -2$

28) $a = 5,\ b = \frac{11}{2}$ 29) $x = 2,\ y = 1$ 30) $x = -8,\ y = -4$

Page 46,47:

1) $a^{3n + 2}$ 2) 2^6 3) $\frac{x^2}{3y}$ 4) 1 5) xy^2 6) $\frac{x^5}{y3}$

7) $\frac{x^2}{y^2}$ 8) $\frac{b^3c^4}{a^4}$ 9) $a^{2k}\ x^{k-1}$ 10) $x^{n-11}y^{n + 7}$

11) $\frac{4x^2}{y^{16}}$ 12) $\frac{1}{y^4}$ 13) $\frac{1}{64}$ 14) 16 15) $\frac{1}{2}$ 16) $2\sqrt{2}$

17) $\frac{1}{9}$ 18) 6 19) $\frac{20}{(mn)^{1/4}}$ 20) 9 21) ± 64

22) $\frac{1}{27}$ 23) $x = 4$ 24) $x = 5$ 25) $x = -6$

26) 4×10^{-2} 27) 1×10^6 28) 2.5×10^1 29) 1

30) 1×10^8 31) 2×10^4

Page 63:

1) $\{1, 3\}$ 2) $\{8, -4\}$ 3) $\{0, 9\}$ 4) $\{0.25\}$

5) $\{-1, 1.67\}$ 6) $\{1.79, -2.79\}$ 7) $\{4.19, -1.19\}$

8) $\{8.47, -0.47\}$ 9) $\{-0.2\}$ 10) $\{-1.5, 5\}$

11) $\{0.58, -2.58\}$ 12) $\{0.25\}$ 13) $\emptyset$ 14) $\{-3, -6\}$